# SIF5020 Linear Methods 

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## Problem 1

a) $E$ is a normed space (with the induced norm) if $E$ is a (vector) subspace of $\ell^{\infty}$. If $x, y \in E$ then also $\alpha x+\beta y$ has only finitely many terms different from zero since $(\alpha x+\beta y)_{n}=\alpha x_{n}+\beta y_{n}$. Hence $E \subseteq \ell^{\infty}$ is a subspace, and therefore a normed space (with the induced norm). $E$ is not complete since the sequence $\left(x^{(n)}\right)$ where $x^{(n)}=\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0,0, \ldots\right)$ is a Cauchy sequence in $E$ that converges in $\ell^{\infty}$ to $x=\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \notin E$. Here

$$
\left\|x^{(n)}-x\right\|_{\infty}=\sup _{n}\left\{\frac{1}{n+1}, \frac{1}{n+2}, \ldots\right\}=\frac{1}{n+1}
$$

showing that $\lim x^{(n)}=x$ in $\ell^{\infty}$, and in particular $\left(x^{(n)}\right)$ is a Cauchy sequence (in both $\ell^{\infty}$ and $E$ ).
b) $E$ is not an inner product space since the parallelogram law fails in $E$. Consider $x=(1,0,0,0, \ldots)$ and $y=(0,1,0,0, \ldots)$. Then $\|x\|=\|y\|=\|x+y\|=\|x-y\|=1$, and

$$
\|x+y\|^{2}+\|x-y\|^{2} \neq 2\left(\|x\|^{2}+\|y\|^{2}\right)
$$

## Problem 2

a) Let $T: C[0,1] \rightarrow C[0,1]$ be given by

$$
(T x)(t)=\int_{0}^{1} t s x(s) d s
$$

We then find that

$$
|T x(t)| \leq t \int_{0}^{1} s\|x\|_{\infty} d s=\frac{t}{2}\|x\|_{\infty}
$$

such that $\|T x\|_{\infty} \leq \frac{1}{2}\|x\|_{\infty}$. Furthermore, if we define $x_{0}(t)=1$ for all $t \in[0,1]$, we can calculate

$$
\left\|T x_{0}\right\|_{\infty}=\max _{0 \leq t \leq 1} t \int_{0}^{1} s d s=\frac{1}{2}\left\|x_{0}\right\|_{\infty}
$$

Together, this shows that $\|T\|=\frac{1}{2}$.
b) Let $T: L^{2}(0,1) \rightarrow L^{2}(0,1)$ be given by

$$
(T x)(t)=\int_{0}^{1} t s x(s) d s
$$

Then by Cauchy-Schwarz

$$
|T x(t)|^{2}=t^{2}\left|\int_{0}^{1} s x(s) d s\right|^{2} \leq t^{2}\left(\int_{0}^{1} s^{2} d s\right)\|x\|_{2}^{2}=\frac{1}{3} t^{2}\|x\|_{2}^{2} .
$$

This gives

$$
\|T x\|^{2}=\int_{0}^{1}|(T x)(t)|^{2} d t \leq\left(\frac{1}{3}\right)^{2}\|x\|_{2}^{2},
$$

and it follows that $\|T\| \leq \frac{1}{3}$.
c) We are looking for a continuous function $x$ on the interval $[0,1]$ that satisfies the integral equation

$$
x(t)=4+\int_{0}^{1} t s x(s) d s ; \quad 0 \leq t \leq 1 .
$$

Let $x=4+T x$ as in a). Then $(I-T) x=4$, or

$$
x=(I-T)^{-1}(4)=\left(\sum_{n=0}^{\infty} T^{n}\right)(4)=\sum_{n=0}^{\infty} T^{n}(4) .
$$

Furthermore,

$$
\begin{aligned}
T(4) & =t \int_{0}^{1} 4 s d s=2 t \\
T(T(4)) & =t \int_{0}^{1} s \cdot 2 s d s=2 t \int_{0}^{1} s^{2} d s=\frac{2 t}{3} \\
& \vdots \\
T^{n}(4) & =\frac{1}{3} T^{n-1}(4), \quad n \geq 2,
\end{aligned}
$$

such that $x=4+2 t\left(1+\frac{1}{3}+\frac{1}{3^{2}}+\ldots\right)=4+3 t$. The problem can alternatively be solved using Banach's Fixed Point Theorem.

## Problem 3

Clearly $\mathcal{N}(B) \subseteq \mathcal{N}(A B)$. Since $A$ has linearly independent columns, $\mathcal{N}(A)=\{0\}$ and we have

$$
(A B) x=0 \quad \Rightarrow \quad A(B x)=0 \quad \Rightarrow \quad B x=0 .
$$

Thus $\mathcal{N}(A B) \subseteq \mathcal{N}(B)$. Hence $\mathcal{N}(A B)=\mathcal{N}(B)$.
Clearly $\mathcal{R}(A B) \subseteq R(A)$. Since $B$ has linearly independent rows, $\mathcal{R}(B)=\mathbb{R}^{r}$ and we have

$$
\begin{aligned}
y \in \mathcal{R}(A) & \Rightarrow y=A z \text { for some } z \in \mathbb{R}^{r} \\
& \Rightarrow y=A B x \text { since } z=B x \text { for some } x \in \mathbb{R}^{n} \\
& \Rightarrow y \in \mathcal{R}(A B) .
\end{aligned}
$$

Thus $\mathcal{R}(A) \subseteq \mathcal{R}(A B)$. Hence $\mathcal{R}(A B)=\mathcal{R}(A)$.

## Problem 4

a) We use Gram-Schmidt on the columns of $A$ (and then normalize):

$$
\begin{aligned}
& b_{(1)}=a_{(1)}=\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right] \\
& b_{(2)}=a_{(2)}-\frac{\left\langle a_{(2)}, b_{(1)}\right\rangle}{\left\langle b_{(1)}, b_{(1)}\right\rangle} b_{(1)}=a_{(2)}-\frac{6}{3} b_{(1)}=\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right] \\
& b_{(3)}=a_{(3)}-\frac{\left\langle a_{(3)}, b_{(1)}\right\rangle}{\left\langle b_{(1)}, b_{(1)}\right\rangle} b_{(1)}-\frac{\left\langle a_{(3)}, b_{(2)}\right\rangle}{\left\langle b_{(2)}, b_{(2)}\right\rangle} b_{(2)}=a_{(3)}-\frac{14}{3} b_{(1)}-\frac{8}{2} b_{(2)}=\frac{1}{3}\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right] \\
& \quad q_{(1)}=\frac{1}{\sqrt{3}}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right], \quad q_{(2)}=\frac{1}{\sqrt{2}}\left[\begin{array}{r}
-1 \\
0 \\
1
\end{array}\right], \quad q_{(3)}=\frac{1}{\sqrt{6}}\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right] .
\end{aligned}
$$

Then

$$
\begin{aligned}
a_{(1)}=b_{(1)} & =\sqrt{3} q_{(1)} \\
a_{(2)}=2 b_{(1)}+b_{(2)} & =2 \sqrt{3} q_{(1)}+\sqrt{2} q_{(2)} \\
a_{(3)}=\frac{14}{3} b_{(1)}+4 b_{(2)}+b_{(3)} & =\frac{14}{3} \sqrt{3} q_{(1)}+4 \sqrt{2} q_{(2)}+\frac{1}{3} \sqrt{6} q_{(3)},
\end{aligned}
$$

and hence

$$
A=\left[\begin{array}{rrr}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{array}\right]\left[\begin{array}{rrr}
\sqrt{3} & 2 \sqrt{3} & \frac{14}{3} \sqrt{3} \\
0 & \sqrt{2} & 4 \sqrt{2} \\
0 & 0 & \frac{1}{3} \sqrt{6}
\end{array}\right] .
$$

b) An orthonormal basis of $U=\operatorname{lin}\{(1,1,1),(1,2,3)\}$ is given by

$$
q_{(1)}=\frac{1}{\sqrt{3}}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right], \quad q_{(2)}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
-1 \\
0 \\
1
\end{array}\right],
$$

and

$$
\begin{aligned}
\operatorname{proj}_{U}\left[\begin{array}{l}
i \\
0 \\
1
\end{array}\right] & =\left\langle\left[\begin{array}{l}
i \\
0 \\
1
\end{array}\right], q_{(1)}\right\rangle q_{(1)}+\left\langle\left[\begin{array}{l}
i \\
0 \\
1
\end{array}\right], q_{(2)}\right\rangle q_{(2)} \\
& =\frac{1}{3}(i+1)\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]+\frac{1}{2}(1-i)\left[\begin{array}{r}
1 \\
0 \\
1
\end{array}\right]=\frac{1}{6}\left[\begin{array}{c}
-1+5 i \\
2+2 i \\
5-i
\end{array}\right] .
\end{aligned}
$$

## Problem 5

a) Since $\left[\begin{array}{rr}-2 & 5 \\ 1 & -1\end{array}\right]^{-1}=\frac{1}{3}\left[\begin{array}{ll}1 & 5 \\ 1 & 2\end{array}\right]$ we have

$$
e^{A}=\frac{1}{3}\left[\begin{array}{ll}
1 & 5 \\
1 & 2
\end{array}\right]\left[\begin{array}{cc}
e^{2} & 0 \\
0 & e^{-1}
\end{array}\right]\left[\begin{array}{rr}
-2 & 5 \\
1 & -1
\end{array}\right]=\frac{1}{3}\left[\begin{array}{cc}
-2 e^{2}+5 e^{-1} & 5\left(e^{2}-e^{-1}\right) \\
-2\left(e^{2}-e^{-1}\right) & 5 e^{2}-2 e^{-1}
\end{array}\right]
$$

b) As $\mathcal{L}(E)$ is a Banach space, we know that "absolute convergence implies convergence". We have

$$
\sum_{n=0}^{\infty} \frac{\left\|T^{n}\right\|}{n!} \leq \sum_{n=0}^{\infty} \frac{\|T\|^{n}}{n!}=e^{\|T\|}
$$

such that $\sum_{n=0}^{\infty} \frac{T^{n}}{n!}$ converges.
$0 \notin \sigma\left(e^{T}\right)$ since $e^{T} e^{-T}=e^{0}=I$ ( $T$ and $(-T)$ commutes).

## Problem 6

The symmetric matrix $A_{n}$ has principal submatrices $A_{1}, \ldots, A_{n}$, and

$$
\operatorname{det} A_{k}=\left|\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & k
\end{array}\right|=\left|\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
0 & & & \\
\vdots & & A_{k-1} & \\
0 & &
\end{array}\right|=\operatorname{det} A_{k-1}
$$

Hence $\operatorname{det} A_{n}=\operatorname{det} A_{n-1}=\cdots=\operatorname{det} A_{1}=1>0$, and $A$ is positive definite.

