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Department of Mathematical Sciences

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Exam in TMA4145 Linear Methods

Friday, August 16, 2013
Time: 09:00 – 13:00
Examination aids: Code D
Grades: Friday, September 6, 2013

No written or handwritten material; calculators Citizen SR-270X (also*College*) or Hewlett Packard HP30S. Each correctly solved problem gives 10 points. Within each problem, subproblems give the same amount of points, except for in problems 2 and 5, where the last subproblem gives one point extra. *All answers shall be thoroughly motivated* *(except for in problem* 1*).*

Problem 1 *(Overview)*
For each of the following, indicate whether it is true or false (no proof required).

(i) $\mathbb{R}^n$ is a Hilbert space ($n \geq 1$).

(ii) The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$ is invertible.

(iii) All Banach spaces are infinite-dimensional.

(iv) A linear transformation between two normed spaces is bounded if and only if it is continuous.

(v) For $A \in M_{n\times n}(\mathbb{R})$ a square matrix the basis of solutions to $\dot{x} = Ax$ is $(n-1)$-dimensional.

(vi) The relation $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ is called the parallelogram identity.

(vii) For any set there is a metric which turns the set into a metric space.

(viii) Lipschitz-continuous functions are continuous.

(ix) The sequence $\{f_n\}_{n\geq 1}$ of functions $f_n \colon x \mapsto \sum_{k=0}^n \frac{x^k}{k!}$ converges in $BC([0,1],\mathbb{R})$, the space of bounded and continuous real-valued functions on $[0,1]$.

(x) The l_p-spaces, $1 \leq p \leq \infty$, are reflexive.

Problem 2 *(Mappings, function spaces and Lipschitz continuity)*

a) Determine the set of constants a such that the matrix

$$A = \begin{bmatrix} 1 & 2 & a \\ 1 & a & 0 \\ 5 & -2 & -1 \end{bmatrix}$$

defines an injective mapping $\mathbb{R}^3 \to \mathbb{R}^3$. For such a, is the same mapping also surjective?

b) Determine whether the function $f\colon x \mapsto \sin(1/x)$ belongs to the space $BC((0,1),\mathbb{R})$ of bounded and continuous real-valued functions on $(0,1)$.

c) Determine whether the function f defined in b) is globally Lipschitz continuous on $(0,1)$.

Problem 3 *(Spectral theory, differential equations)* Let $A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$.

a) Express A in Jordan normal form TJT^{-1}.

b) Find the solution of the initial-value problem

$$\dot{x} = Ax, \qquad x(0) = x_0,$$

for general initial data $x_0 \in \mathbb{R}^2$.

Problem 4 *(Inner products, least distances)*

a) Show that the norm $\|\cdot\|_\infty$ given by $\sup_{t\in\mathbb{R}} |x(t)|$ does not come from an inner product (the norm is defined on all bounded and continuous real-valued functions).

b) In $L^2((-\pi,\pi),\mathbb{C})$ with inner product $\langle x, y\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\overline{y(t)}\,dt$, find the minimal distance between $M = \operatorname{span}\{1, \mathrm{e}^{it}, \mathrm{e}^{2it}\}$ and the function x_0 given by $x_0(t) = t$.

Problem 5 *(Bounded linear transformations, Banach- and Hilbert-space theory)*
For each of the following linear transformations, determine the operator norm and give an example of an element for which the norm is attained (i.e., determine $\|T\|$ and find a non-zero element $x \in X$ such that $\|Tx\|_Y / \|x\|_X = \|T\|$, where $T\colon X \to Y$ is given in the respective assignment).

a) $T\colon l^1(\mathbb{R}) \to l^1(\mathbb{R})$ given by

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

b) $T\colon L^2((-\pi,\pi),\mathbb{R}) \to \mathbb{R}$ given by

$$T(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(x)\, dx.$$

c) $T\colon L^2((-\pi,\pi),\mathbb{C}) \to L^2((-\pi,\pi),\mathbb{C})$ given by

$$\mathcal{T}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \exp(ix\xi)\, dx.$$

Problem 6 *(The Gram–Schmidt orthonormalization process)*

a) Perform a Gram–Schmidt orthonormalization of the column vectors A_1, A_2 and A_3 of

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 6 \\ 3 & 4 & 9 \end{bmatrix}.$$

Give the corresponding QR-decomposition of the matrix A.

b) Let $\{x_1, x_2, \ldots\}$ be a sequence of linearly independent vectors in an inner-product space. Give a rigorous inductive definition of the vectors $\{e_1, e_2, \ldots\}$ constructed in the standard Gram-Schmidt orthonormalization process and prove that $\{e_j\}_j$ is indeed an orthonormal sequence spanning the same closed linear span as $\{x_j\}_j$.