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Department of Mathematics

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English version

TMA4145 Linear Methods: Continuation Exam

16th August 2010
Time: 15:00–19:00
Examination Aids: D
No written and handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Answer any *four* of the following.

i. Give a definition of a continuous function from one metric space to another.
ii. Define what it means for a metric space to be *complete*.
iii. State the *rank theorem* (also called the *rank–nullity theorem*).
iv. In the $PA = LU$ factorisation of a matrix, what properties do the four matrices have?
v. Give a definition of an *orthonormal family* in an inner product space.
vi. Write out the *parallelogram identity* (also called the *parallelogram law*) for the norm in an inner product space.
vii. Define the L^2–norm on $C([0,1],\mathbb{C})$, the space of complex–valued continuous functions on $[0,1]$.

(8 points)

Problem 2.

Define $W \subseteq \mathbb{R}^5$ as the subspace:

$$W := \left\{ \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} \quad \text{with} \quad \begin{array}{rcrcrcrcrl} v & + & 3w & - & x & + & 3y & + & 5z & = 0, \\ 3v & + & 9w & - & 3x & + & 3y & + & 9z & = 0, \\ 2v & + & 6w & - & 2x & & & + & 4z & = 0, \\ v & + & 3w & - & x & - & 3y & - & z & = 0 \end{array} \right\}$$

Find the closest point in W to the vector:

$$\begin{bmatrix} 4 \\ 4 \\ -4 \\ -1 \\ 5 \end{bmatrix}$$

(8 points)

Problem 3.

Let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{R}^2$. Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the function defined by:

$$d(\vec{x}, \vec{y}) := \begin{cases} \|\vec{x} - \vec{y}\|_2 & \text{if } \mu\vec{x} = \lambda\vec{y} \text{ for some } \mu, \lambda \in \mathbb{R}, \\ \|\vec{x}\|_2 + \|\vec{y}\|_2 & \text{otherwise.} \end{cases}$$

1. Prove that d defines a metric on $\mathbb{R}^2$. *(4 points)*
2. Does d come from a norm? Justify your answer. *(2 points)*
3. Let $I\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the identity map. Is either of $I\colon (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, d)$ or $I\colon (\mathbb{R}^2, d) \to (\mathbb{R}^2, \|\cdot\|_2)$ continuous? Justify your answer. *(2 points)*

Problem 4.

Let Poly_1 be the space of polynomials of degree at most 1 with real coefficients. Define an inner product on Poly_1 by the formula:

$$\langle p, q \rangle := p(0)q(0) + p(1)q(1).$$

You may assume that this does define an inner product.

1. Define a linear function $\mathrm{Poly}_1 \to \mathbb{R}$ by $p(t) \mapsto \int_0^1 p(t)dt$. Find a polynomial $r(t) \in \mathrm{Poly}_1$ such that:
$$\langle p(t), r(t) \rangle = \int_0^1 p(t)dt$$
for all polynomials $p(t) \in \mathrm{Poly}_1$. *(2 points)*
2. Let $D\colon \mathrm{Poly}_1 \to \mathrm{Poly}_1$ be the differentiation operator: $(Dp)(t) = p'(t)$. Find its *adjoint*. That is, find the operator $D^*\colon \mathrm{Poly}_1 \to \mathrm{Poly}_1$ which satisfies:
$$\langle Dp(t), q(t) \rangle = \langle p(t), D^* q(t) \rangle$$
for all $p(t), q(t) \in \mathrm{Poly}_1$. *(4 points)*
3. Find a polynomial $s(t) \in \mathrm{Poly}_1$ such that for all $p(t) \in \mathrm{Poly}_1$ $\langle p(t), s(t) \rangle = \int_0^1 Dp(t)dt$. *(2 points)*

Problem 5.

Let A be the matrix

$$\begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.875 \end{bmatrix}$$

1. Explain why A has a unique eigenvector in the positive quadrant with length 1.

   That is, there is a unique vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x, y \geq 0$ and $x^2 + y^2 = 1$ which is an eigenvector of A.

   Note: you are not required to actually find this eigenvector. *(3 points)*
2. Show that if $\vec{x}$ is a vector in the positive quadrant (i.e. both components positive) then $A\vec{x}$ is also in the positive quadrant.

   What does this tell you about the eigenvalue of A corresponding to the eigenvector referred to in part 1? *(2 points)*
3. Explain how one could use Banach's Fixed Point Theorem to find the eigenvector referred to in part 1.

   Note: this question is asking you to *set up* the problem. You do not have to check that all the conditions apply, but you should explain what the conditions are for this particular case, and you should note which conditions have already been verified by the previous steps.

   If, in your answer to part 1, you actually found the eigenvector, for this part pretend that you do not know it. *(3 points)*