NTNU

Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods**

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Examination date: Wednesday, December 11, 2013
Examination time (from–to): 15:00–19:00
Permitted examination support material: Code D: No written or handwritten material, calculators Citizen SR-270X (including the College version) or Hewlett Packard HP30S.

Other information:
As in the course, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Unless otherwise stated, it is to be assumed that the standard basis, inner product, norm and distance in $\mathbb{R}^n$ and $\mathbb{C}^n$ are to be used. Except Problem 1, all solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified.

The exam contains 11 parts (Problem 1 counts as one part). Each solution will be judged as *rudimentary*, *acceptable*, *good* or *excellent*. Five acceptable solutions guarantee an E; seven acceptable with at least one good a D, seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

Language: English
Number of pages: 9
Number pages enclosed: 0

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Problem 1 (Overview)
For each of the following, state whether it is true or false (no proof required).

(i) There exists a bijective function $\mathbb{Q} \to \mathbb{N}$.
Yes. See Exercise set 1.

(ii) The l_p-spaces, $1 \leq p \leq \infty$, are all Hilbert spaces.
No. Only l_2 can be given an inner product that is compatible with the l_p-norm.

(iii) All linear transformations $\mathbb{R}^n \to \mathbb{R}^m$, with $n, m \in \mathbb{N}$, can be realised by matrices.
Yes. This is done by choosing a basis for each space.

(iv) The function $t \mapsto \sin(1/t)$ lies in the closed unit ball in $BC((0,1),\mathbb{R})$ (endowed with the standard supremum norm).
Yes. It is continuous on $(0,1)$ (although not on $[0,1]$), and bounded with $\|t \mapsto \sin(1/t)\|_{BC((0,1),\mathbb{R})} = \sup_{t\in(0,1)} |\sin(1/t)| = 1$.

(v) The rank of matrix is always the same as the dimension of its null space.
No. The sum of the rank and the dimension of the nullspace equals the dimension of the domain of definition.

(vi) For any orthonormal sequence of vectors $\{e_j\}_j$ in a Hilbert space, and any sequence $\{c_j\}_j \in l_2$ of scalars, one has $\langle \sum_j c_j e_j, \sum_k c_k e_k \rangle = \sum_j |c_j|^2$.
Yes. $\langle \sum_j c_j e_j, \sum_k c_k e_k \rangle = \sum_{j,k} c_j \overline{c_k} \langle e_j, e_k \rangle = \sum_j |c_k|^2$.

(vii) The Cauchy–Schwarz inequality is valid in any Banach space.
No. The Cauchy–Schwarz inequality $|\langle x, y\rangle| \leq \|x\|\|y\|$ requires an inner-product space.

(viii) The set $\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + 2x_2^2 \leq 1\}$ is convex.
Yes. It is an ellipse.

(ix) $L_2((-\pi,\pi),\mathbb{R})$ is isometrically isomorphic to its dual.
Yes. This is the meaning of the Riesz representation theorem.

(x) The initial-value problem $\dot{x} = \sqrt{x}$, $x(0) = 0$, has a unique solution $u \in C^1([0,\infty),\mathbb{R})$.
No. The right-hand side is not Lipschitz, and one easily checks that $x(t) = 0$ and $x(t) = \frac{1}{4}t^2$ both satisfy this initial-value problem (no uniqueness).

Problem 2 (Linear transformations)
This problem is meant to test knowledge of definitions and basic manipulation and calculation abilities.

a) Determine the range of the matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$. Is A invertible; self-adjoint; nilpotent; unitary? For each concept, provide the definition together with your answer.

Invertibility means that there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Since $A^2 = I$, we have $A = A^{-1}$, and A is invertible.

Self-adjointness means $\langle Ax, y\rangle = \langle x, Ay\rangle$ which for a real $n \times n$-matrix means that A is symmetric ($a_{ij} = a_{ji}$). Hence A is self-adjoint.

The matrix A is nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$. Since $A^{2k} = I$, for all natural numbers k, this is impossible. A is not nilpotent.

A is unitary if $AA^* = A^*A = I$. In view of that $A^* = A = A^{-1}$, A is unitary.

b) What is the operator norm of A?

Unitary maps are isometries, meaning $\|Ax\|_{\mathbb{R}^2} = \|x\|_{\mathbb{R}^2}$ for all $x \in \mathbb{R}^2$. The operator norm of A is thus 1.

A more direct way to see this is that

$$\begin{aligned}
\|Ax\|_{\mathbb{R}^2} &= \left\| \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \right\|_{\mathbb{R}^2} \\
&= \left(\frac{1}{2} \left(x_1^2 + 2x_1x_2 + x_2^2 + x_1^2 - 2x_1x_2 + x_2^2 \right) \right)^{1/2} \\
&= \left(x_1^2 + x_2^2 \right)^{1/2} \\
&= \|x\|_{\mathbb{R}^2}.
\end{aligned}$$

c) Given that $\cosh(t) = \frac{e^t + e^{-t}}{2}$ and $\sinh(t) = \frac{e^t - e^{-t}}{2}$, what is $\exp(tA)$?

Note first that

$$\cosh(t) = \frac{e^t + e^{-t}}{2} = \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} + \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!},$$

and

$$\sinh(t) = \frac{e^t - e^{-t}}{2} = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} - \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}.$$

In view of the identity $A^2 = I$, we then find that

$$\begin{aligned}
\exp(tA) &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{t^{2k}A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1}A^{2k+1}}{(2k+1)!} \\
&= \left(\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right) I + \left(\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\right) A \\
&= \cosh(t) I + \sinh(t) A.
\end{aligned}$$

Problem 3 *(Metric spaces)*
Let d be the distance on $\mathbb{R}$ given by

$$d(x,y) = \frac{1}{\pi}\left|\arctan(x) - \arctan(y)\right|.$$

a) Verify that d is a metric on $\mathbb{R}$.

Finiteness and positivity: The function arctan *is well-defined* $\mathbb{R} \to (-\pi/2, \pi/2)$*, so the triangle inequality for real numbers guarantees that d is well-defined with values in* $[0,1) \subset [0,\infty)$*:*

$$\begin{aligned} 0 &\leq \frac{1}{\pi}\left|\arctan(x) - \arctan(y)\right| \\ &\leq \frac{1}{\pi}\left(|\arctan(x)| + |\arctan(y)|\right) \\ &< \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= 1. \end{aligned}$$

Symmetry: This follows from the symmetry of $|a-b|$ for real numbers a and b:

$$d(x,y) = \frac{1}{\pi}\left|\arctan(x) - \arctan(y)\right| = \frac{1}{\pi}\left|\arctan(y) - \arctan(x)\right| = d(y,x).$$

Triangle inequality: This similarly follows from the triangle inequality for real numbers:

$$\begin{aligned} d(x,y) &= \frac{1}{\pi}\left|\arctan(x) - \arctan(y)\right| \\ &\leq \frac{1}{\pi}\Big(\left|\arctan(y) - \arctan(z)\right| + \left|\arctan(z) - \arctan(x)\right|\Big) \\ &= d(y,z) + d(z,x). \end{aligned}$$

Non-degeneracy:

$$d(x,y) = 0 \iff \arctan(x) = \arctan(y) \iff x = y,$$

since arctan *is injective.*

b) Show that the open unit ball in $(\mathbb{R}, d)$ is also closed, and that $(\mathbb{R}, d)$ is not a complete metric space.

Closedness: According to the definition of an open unit ball in a metric space with a zero element, we have

$$B_1(0) = \{x \in \mathbb{R} \colon d(x,0) < 1\}.$$

For the metric d,

$$\begin{aligned} d(x,0) < 1 \quad &\iff \quad \frac{1}{\pi}\,|\arctan(x) - \arctan(0)| < 1 \\ &\iff \quad |\arctan(x)| < \pi \\ &\iff \quad x \in \mathbb{R}. \end{aligned}$$

Hence, $B_1(0) = \mathbb{R}$ coincides with the whole space, which by definition is closed.

Incompleteness: Consider $x_n = n$, $n \in \mathbb{N}$. Then

$$\begin{aligned} \lim_{m,n\to\infty} d(x_n, x_m) &= \frac{1}{\pi} \lim_{m,n\to\infty} |\arctan(n) - \arctan(m)| \\ &= \frac{1}{\pi} |\lim_{n\to\infty} \arctan(n) - \lim_{m\to\infty} \arctan(m)| \\ &= \frac{1}{\pi} \left| \frac{\pi}{2} - \frac{\pi}{2} \right| \\ &= 0, \end{aligned}$$

since $|\cdot|$ is continuous (this enables moving the limits inside the absolute value), and the limit $\lim_{j\to\infty} \arctan(j)$ exists (this enables separating the two limits). Hence, $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{R}, d)$.

By the same argument, however,

$$d(x_n, 0) \to 1 \quad \text{as} \quad n \to \infty,$$

so that any limit x must satisfy $d(x,0) = 1$. Since there are no such $x \in \mathbb{R}$, we have found a non-convergent Cauchy sequence in $(\mathbb{R}, d)$. Thus, $(\mathbb{R}, d)$ is incomplete.

Problem 4 *(Spectral theory)*

a) A 2×2 symmetric matrix has an eigenvalue 4 with eigenvector $(1, 2)$. The matrix also has an eigenvalue 1. Use this to determine the matrix.

The spectral theorem guarantees that the matrix has an orthonormal basis of eigenvectors, such that

$$A = QDQ^t,$$

where D is the diagonal matrix of eigenvalues and Q is the orthogonal matrix of corresponding eigenvectors (in the same order as the eigenvalues). With

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{Q} = \begin{bmatrix} 1 & v_1 \\ 2 & v_2 \end{bmatrix},$$

we thus only need to determine the matrix $\tilde{Q}$ such that $v = (v_1, v_2) \perp (1, 2)$, and then to normalise the length of the vectors. A solution of this is

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Calculating $QDQ^t = QDQ$ (note that Q is also symmetric) gives us

$$A = \frac{1}{5} \begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix}.$$

b) Express

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

in Jordan normal form, determining both the matrix J and the change-of-basis matrix T in $A = TJT^{-1}$. *Hint: this matrix has an eigenvalue of triple algebraic multiplicity.*

Characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & -1 \\ 0 & 1 & 2-\lambda \end{bmatrix} \\ &= -\lambda^3 + 9\lambda^2 - 27\lambda + 27 \\ &= (3-\lambda)^3. \end{aligned}$$

Eigenvectors:

$$\ker(A-3I) = \ker \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Generalised Eigenvectors: Since $\ker(A-3I)$ *is already two-dimensional, we know directly that* $\ker(A-3I)^2 = \mathbb{R}^3$ *(the generalised eigenspaces are strictly increasing), and we can pick any vector not in* $\ker(A-3I)$ *to obtain a generalised eigenvector. For example, pick*

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \ker(A-3I)^2 \setminus \ker(A-3I),$$

and let

$$v_1 = (A-3I)v_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \in \ker(A-3I).$$

These constitute a Jordan chain.

We still need to add a vector from $\ker(A-3I)$*, linearly independent from* $\{v_1, v_2\}$*, but that we already have. Let*

$$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then the change-of-basis matrix

$$T = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

corresponds to the Jordan norm form

$$J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

of the matrix A*.*

Problem 5 *(Inner products, Hilbert spaces)*

a) In $\mathbb{R}^4$, let

$$M = \operatorname{span}\{(1,2,3,2),(1,0,1,1)\} \quad \text{and} \quad y = (2,0,3,1).$$

Calculate $d = \operatorname{dist}(y, M)$. Is there a point $x_0 \in M$ with $\|x_0 - y\| = d$ (if so, explain why and determine it; if not, justify why there cannot be one)?

Gram–Schmidt: Let

$$e_1 = \frac{(1,0,1,1)}{\|(1,0,1,1)\|} = \frac{1}{\sqrt{3}}(1,0,1,1).$$

Then let

$$\begin{aligned} \tilde{v}_2 &= (1,2,3,2) - \left\langle (1,2,3,2), \tfrac{1}{\sqrt{3}}(1,0,1,1) \right\rangle \tfrac{1}{\sqrt{3}}(1,0,1,1) \\ &= (1,2,3,2) - (2,0,2,2) \\ &= (-1,2,1,0), \end{aligned}$$

and

$$e_2 = \frac{(-1,2,1,0)}{\|(-1,2,1,0)\|} = \frac{1}{\sqrt{6}}(-1,2,1,0).$$

Then $\{e_1, e_2\}$ *form an orthonormal set, spanning the same linear space as* $\{(1,2,3,2),(1,0,1,1)\}$. *Note that this set is closed, convex and non-empty, so that the existence of a minimising point* x_0*, as asked for the in the problem, is guaranteed by the minimal distance theorem.*

To find it, project y *onto* $\operatorname{span}\{e_1, e_2\}$*:*

$$\begin{aligned} x_0 &= \langle y, e_1\rangle e_1 - \langle y, e_2\rangle e_2 \\ &= \langle (2,0,3,1), \tfrac{1}{\sqrt{3}}(1,0,1,1)\rangle \tfrac{1}{\sqrt{3}}(1,0,1,1) \\ &\quad + \langle (2,0,3,1), \tfrac{1}{\sqrt{6}}(-1,2,1,0)\rangle \tfrac{1}{\sqrt{6}}(-1,2,1,0) \\ &= (2,0,2,2) + \tfrac{1}{6}(-1,2,1,0) \\ &= \tfrac{1}{6}(11,2,13,12). \end{aligned}$$

The distance d *is then given by*

$$d^2 = \|y\|^2 - |\langle y, e_1\rangle|^2 - |\langle y, e_2\rangle|^2 = 14 - 12 - \frac{1}{6} = \frac{11}{6},$$

meaning $d = \sqrt{\frac{11}{6}}$.

b) Prove that there is $c \geq 0$ such that

$$\int_{-\pi}^{\pi} x(t)\sin(2t)\,dt \leq c\left(\int_{-\pi}^{\pi} |x(t)|^2\,dt\right)^{1/2},$$

for any $x \in C([-\pi,\pi],\mathbb{R})$, and that the choice $c = \sqrt{\pi}$ is the least possible.

The set $C([-\pi,\pi],\mathbb{R})$ is a linear subspace of $L_2((-\pi,\pi),\mathbb{R})$, and for any $x \in L_2((-\pi,\pi),\mathbb{R})$, we have

$$|\langle x, \sin(2\cdot)\rangle| \leq \|\sin(2\cdot)\|\|x\|, \qquad (**)$$

according to the Cauchy–Schwarz inequality. Here $\langle x, y\rangle = \int_{-\pi}^{\pi} x(t)y(t)\,dt$ and $\|x\| = \langle x, x\rangle^{1/2}$ denote the inner product and corresponding norm on $L_2((-\pi,\pi),\mathbb{R})$.

Now, $(\sin(s))^2 = \frac{1}{2} - \frac{\cos(2s)}{2}$ has mean $\frac{1}{2}$ over any period of $\cos(2\cdot)$, so

$$\|\sin(2\cdot)\| = \left(\int_{-\pi}^{\pi} (\sin(2t))^2\,dt\right)^{1/2} = \sqrt{\pi}.$$

But this means $(**)$ is $(*)$ with $c = \sqrt{\pi}$. Since we know that we have equality in Cauchy–Schwarz for linearly dependent vectors, the choice $x = \sin(2\cdot) \in C([-\pi,\pi],\mathbb{R})$ yields that there can be no smaller constant c.

c) Let H be a Hilbert space (real or complex) with inner product $\langle\cdot,\cdot\rangle$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$. Given an element $y \in H$, show that the mapping $x \mapsto \sum_{j\in\mathbb{N}}\langle x, e_j\rangle\langle e_j, y\rangle$ is a bounded linear functional on H.

We know that, by the Riesz representation theorem, all bounded linear functionals on H are of the form

$$x \mapsto \langle x, z\rangle,$$

for some $z \in H$. So a qualified guess is that $z = y$.

To prove this, note that since $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for H, we have

$$\begin{aligned}
\langle x, y\rangle &= \left\langle \sum_{j\in\mathbb{N}}\langle x, e_j\rangle e_j, \sum_{k\in\mathbb{N}}\langle y, e_k\rangle e_k \right\rangle \\
&= \sum_{j,k\in\mathbb{N}} \langle x, e_j\rangle\overline{\langle y, e_k\rangle}\langle e_j, e_k\rangle \\
&= \sum_{j\in\mathbb{N}}\langle x, e_j\rangle\overline{\langle y, e_j\rangle} \\
&= \sum_{j\in\mathbb{N}}\langle x, e_j\rangle\langle e_j, y\rangle.
\end{aligned}$$

Hence, the mapping considered in the problem is the mapping $x \mapsto \langle x, y\rangle$, which, by the Riesz representation theorem, is a bounded linear functional on H.