Department of Mathematical Sciences

Examination paper for **TMA4145 Linear Methods**

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Examination date: Wednesday, December 11, 2013
Examination time (from–to): 15:00–19:00
Permitted examination support material: Code D: No written or handwritten material. Calculators Citizen SR-270X (including the College version) or Hewlett Packard HP30S.

Other information:
As in the course, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Unless otherwise stated, it is to be assumed that the standard basis, inner product, norm and distance in $\mathbb{R}^n$ and $\mathbb{C}^n$ are to be used. Except Problem 1, all solutions should be stated in a precise and rigorous way, with any assumptions written down and arguments justified.

The exam contains 11 questions (Problem 1 is seen as one question). Each solution will be judged as *rudimentary*, *acceptable*, *good* or *excellent*. Five acceptable solutions guarantee an E; seven acceptable with at least one good a D, seven acceptable with at least five good a C; nine good with at least two excellent a B; nine good with at least seven excellent an A. These are guaranteed limits. Beyond that, the grade is based on the total achievement.

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Date Signature

Problem 1 (Overview)
For each of the following, state whether it is true or false (no proof required).

(i) There exists a bijective function $\mathbb{Q} \to \mathbb{N}$.

(ii) The l_p-spaces, $1 \le p \le \infty$, are all Hilbert spaces.

(iii) All linear transformations $\mathbb{R}^n \to \mathbb{R}^m$, with $n, m \in \mathbb{N}$, can be realised by matrices.

(iv) The function $t \mapsto \sin(1/t)$ lies in the closed unit ball in $BC((0,1),\mathbb{R})$ (endowed with the standard supremum norm).

(v) The rank of a matrix is always the same as the dimension of its null space.

(vi) For any orthonormal sequence of vectors $\{e_j\}_j$ in a Hilbert space, and any sequence $\{c_j\}_j \in l_2$ of scalars, one has $\langle \sum_j c_j e_j, \sum_k c_k e_k \rangle = \sum_j |c_j|^2$.

(vii) The Cauchy–Schwarz inequality is valid in any Banach space.

(viii) The set $\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + 2x_2^2 \le 1\}$ is convex.

(ix) $L_2((-\pi,\pi),\mathbb{R})$ is isometrically isomorphic to its dual.

(x) The initial-value problem $\dot{x} = \sqrt{x}$, $x(0) = 0$, has a unique solution $u \in C^1([0,\infty),\mathbb{R})$.

Problem 2 (Linear transformations)
This problem is meant to test knowledge of definitions and basic manipulation and calculation abilities.

a) Determine the range of the matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$. Is A invertible; self-adjoint; nilpotent; unitary? For each concept, provide the definition together with your answer.

b) What is the operator norm of A?

c) Given that $\cosh(t) = \frac{e^t + e^{-t}}{2}$ and $\sinh(t) = \frac{e^t - e^{-t}}{2}$, what is $\exp(tA)$?

Problem 3 (Metric spaces)
Let d be the distance on $\mathbb{R}$ given by

$$d(x,y) = \frac{1}{\pi}\left|\arctan(x) - \arctan(y)\right|.$$

a) Verify that d is a metric on $\mathbb{R}$.

b) Show that the open unit ball in $(\mathbb{R}, d)$ is also closed, and that $(\mathbb{R}, d)$ is not a complete metric space.

Problem 4 (Spectral theory)

a) A 2×2 symmetric matrix has an eigenvalue 4 with eigenvector $(1, 2)$. The matrix also has an eigenvalue 1. Use this to determine the matrix.

b) Express

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

in Jordan normal form, determining both the matrix J and the change-of-basis matrix T in $A = TJT^{-1}$. *Hint: this matrix has an eigenvalue of triple algebraic multiplicity.*

Problem 5 (Inner products, Hilbert spaces)

a) In $\mathbb{R}^4$, let

$$M = \operatorname{span}\{(1,2,3,2),(1,0,1,1)\} \quad \text{and} \quad y = (2,0,3,1).$$

Calculate $d = \operatorname{dist}(y, M)$. Is there a point $x_0 \in M$ with $\|x_0 - y\| = d$ (if so, explain why and determine it; if not, justify why there cannot be one)?

b) Prove that there is $c \geq 0$ such that

$$\int_{-\pi}^{\pi} x(t)\sin(2t)\,dt \leq c\left(\int_{-\pi}^{\pi} |x(t)|^2\,dt\right)^{1/2},$$

for any $x \in C([-\pi,\pi],\mathbb{R})$, and that the choice $c = \sqrt{\pi}$ is the least possible.

c) Let H be a Hilbert space (real or complex) with inner product $\langle\cdot,\cdot\rangle$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$. Given an element $y \in H$, show that the mapping $x \mapsto \sum_{j\in\mathbb{N}}\langle x, e_j\rangle\langle e_j, y\rangle$ is a bounded linear functional on H.