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Department of Mathematics

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English version

TMA4145 Linear Methods: Final Exam

Thursday 17th December 2009
Time: 09:00–13:00
Examination Aids: D
No written and handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Answer any **four** of the following.

i. Give the definition of a metric (function) on a set.
ii. Give the definition of a convergent sequence in a metric space.
iii. State Banach's Fixed Point Theorem.
iv. State the Spectral Theorem in finite dimensions.
v. Define a neighbourhood of a point in a metric space.
vi. State the Cauchy–Schwarz inequality.
vii. In the QR–factorisation of a matrix, what properties do Q and R have?

(8 points)

Problem 2.

Let (M, d) be a metric space.

1. Let (x_n) be a sequence in M converging to a point, say $x \in M$. Show that there is a subsequence (x_{n_k}) of (x_n) such that $d(x, x_{n_k}) < \frac{1}{k}$ for all $k \in \mathbb{N}$. *(2 points)*
2. As before, let (x_n) be a sequence in M converging to a point, say $x \in M$. For each n, let $(x_n^m)_{m \in \mathbb{N}}$ be a sequence in M converging (as $m \to \infty$) to x_n. By applying part 1 to all of these sequences, or otherwise, prove that there is a sequence (y_k) such that
   (a) $(y_k) \to x$
   (b) for each $k \in \mathbb{N}$, there exist $n_k, m_k \in \mathbb{N}$ such that $y_k = x_{n_k}^{m_k}$
   (c) the n_k and m_k can be chosen such that $n_k > n_{k-1}$ and $m_k > m_{k-1}$.

   (3 points)
3. Let $A \subseteq M$ be a subset. An accumulation point of A is a point $x \in M$ (not necessarily in A) for which there is a sequence in A converging to x. Show that if (x_n) is a sequence of accumulation points of A which converges in M then its limit is also an accumulation point of A. *(2 points)*
4. In $(C([0,1], \mathbb{C}), \|\cdot\|_\infty)$ the following can be shown:
   (a) Every continuous function is the limit of a sequence of piecewise linear functions,
   (b) Every piecewise linear function is the limit of a sequence of polynomials.

   Explain why, for a continuous function $f\colon [0,1] \to \mathbb{C}$ and $\epsilon > 0$, there is a polynomial p such that $|f(t) - p(t)| < \epsilon$ for all $t \in [0,1]$. *(1 point)*

Problem 3.

A matrix A has QR–factorisation:

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 8 & -6 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1. Explain why $A^\top A = R^\top R$ and compute this matrix. *(2 points)*
2. Find an orthonormal basis of $\mathbb{R}^3$ of eigenvectors of $R^\top R$. *(3 points)*
3. Find the singular value decomposition of R and hence find the singular value decomposition of A. *(3 points)*

Problem 4.

For $k \in \mathbb{N}$, let Poly_k be the vector space of polynomials with complex coefficients of degree at most k. Define a function $\mathrm{Poly}_3 \times \mathrm{Poly}_3 \to \mathbb{C}$ by

$$\langle p, q \rangle = \sum_{j=0}^{3} p(j)\overline{q(j)}$$

1. Prove that this is an inner product on Poly_3. *(3 points)*
2. Apply the Gram–Schmidt algorithm to the family $\{1, t\}$ to find an orthonormal basis for Poly_1, regarded as a subspace of Poly_3 in the obvious way. *(2 points)*
3. Let $p \in \mathrm{Poly}_3$ be a polynomial such that $p(0) = -3$, $p(1) = 1$, $p(2) = 3$, and $p(3) = 3$. Find the orthogonal projection of p on to Poly_1 (that is, find the closest point to p in the subspace Poly_1). *(2 points)*
4. Is $\deg p \leq 1$? *(1 point)*

Problem 5.

The Newton–Raphson method for finding roots of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ is to iterate the function

$$x \mapsto g(x) := x - \frac{f(x)}{f'(x)}.$$

We assume the following conditions on f:

i. f has continuous second derivative,

ii. $f'(x) \neq 0$ for all $x \in \mathbb{R}$,

iii. there is some $0 < \alpha < 1$ such that $|f(x)f''(x)| \leq \alpha |f'(x)|^2$ for all $x \in \mathbb{R}$.

1. Show that x^* is a fixed point of g if and only if $f(x^*) = 0$. *(1 point)*
2. Prove that g is a contraction. *(2 points)*
   You may find it useful to remember the Mean Value Theorem: that for a differentiable function $h\colon \mathbb{R} \to \mathbb{R}$ and $x < y \in \mathbb{R}$ then there is some $z \in (x, y)$ such that
   $$h(y) - h(x) = (y - x)h'(z).$$
3. Does this proceedure work with the polynomial $f(x) = x^k$ ($k \in \mathbb{N}$)? If not, which of the conditions fail? *(2 points)*
4. The proceedure does work for the polynomial $f(x) = x^3 + 3x + 1$ with $\alpha = 8/9$. Using the estimate $|x_n - x^*| \leq \alpha^n |x_0 - x_1|$, estimate how many iterations would be needed to find x^* to within .01 starting with $x_0 = 0$. *(1 point)*
5. Do 5 iterations starting with $x_0 = 0$, recording them to an appropriate degree of accuracy. What do you observe? *(2 points)*