Problem 1

a) *State (without proof) whether the assertion is true or false. Find a counterexample if it is false.*

1. Any matrix $A \in \mathcal{M}_{2\times 2}(\mathbb{C})$ is diagonalizable. False, for example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalizable.

2. A Cauchy sequence in a Banach space is always convergent. True

3. Norms on $\mathbb{R}^n$ are equivalent. True

4. A finite dimensional inner product space is always complete. True

5. A continuous linear operator on a normed space is always bounded. True

b) *Definitions and theorems.*

1. Define the notion of a Cauchy sequence in a metric space (X, d). $\{x_n\}$ is Cauchy means that

$$\sup_{n,m\geq N} d(x_n, x_m) \to 0, \quad \text{as } N \to \infty.$$

2. Define the adjoint of a bounded linear operator T between two Hilbert spaces $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$. The adjoint, say T^*, of T is determined by

$$\langle Tu, v\rangle_Y = \langle u, T^*v\rangle_X, \quad \text{for all } u \in X, v \in Y.$$

3. Define the orthogonal complement of a subspace M of a Hilbert space H. The orthogonal complement of a subspace M of a Hilbert space H is defined by

$$M^\perp := \{h \in H : \langle h, m\rangle = 0, \text{ for all } m \in M\}.$$

4. State the Banach fixed point theorem (without proof). Let T be a mapping from a complete metric space (X, d) to itself. Assume that there is a constant $0 \leq L < 1$ such that

$$d(Tx, Ty) \leq L\, d(x, y)$$

for all $x, y \in X$. Then there exists a unique point, say $\hat{x} \in X$, such that $T\hat{x} = \hat{x}$.

5. State the Riesz representation theorem (without proof). Let f be a bounded linear functional on a Hilbert space H, then there exists a unique $u \in H$ such that

$$f(h) = \langle h, u\rangle, \quad \text{for all } h \in H.$$

Moreover, $||f|| = ||u||$ (If you did not write this final norm equality in the exam paper, it is also OK).

Problem 2

Let $A \in \mathcal{M}_{4\times 4}(\mathbb{C})$. Assume that 1 is the only eigenvalue of A. Write down all the possible Jordan normal forms (up to similarity) of A. They are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 3

Let

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

a) Determine if A is normal and find the singular values of A. Note that AA^* is 3×3, but A^*A is 2×2, so they can never equal, hence A is not normal. To compute singular values of A, we need to compute the eigenvalue of

$$A^*A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

the idea is to compute $\det(\lambda - A^*A)$, which is

$$\det \begin{pmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 2 \end{pmatrix} = \lambda^2 - 4\lambda.$$

So the positive eigenvalue of A^*A is 4. Hence 2 is the singular value of A.

b) Find a singular value decomposition of A. First, we need to (unitarily) diagnonalize A^*A. By a direct computation, we know that

$$e_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e_4 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are unit eigenvectors of A^*A with eigenvalue $0, 4$ respectively, put

$$V = (e_4, e_0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we get

$$V^*A^*AV = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

To compute U, note that

$$AV = (Ae_4, 0),$$

so U is given by a unitary matrix of the form $(Ae_4/||Ae_4||, U_2, U_3)$, compute

$$Ae_4/||Ae_4|| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},$$

so we can choose

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\0 & \sqrt{2} & 0\\-1 & 0 & 1\end{pmatrix},$$

put

$$\Sigma = \begin{pmatrix}2 & 0\\0 & 0\\0 & 0\end{pmatrix},$$

we know that $A = U\Sigma V^*$ with U, V, Σ above is the singular value decomposition of A.

c) Find $\hat{x} = \begin{pmatrix}\hat{x}_1\\\hat{x}_2\end{pmatrix}$ with minimal norm such that

$$||A\hat{x} - b|| = \min_{x\in\mathbb{C}^2} ||Ax - b||, \quad \text{for} \quad b = \begin{pmatrix}4\\0\\0\end{pmatrix}.$$

We know that

$$\hat{x} = A^+ b = V\Sigma^+ U^* b = \begin{pmatrix}1\\-1\end{pmatrix}.$$

Problem 4

Apply Picard iteration to

$$x'(t) = 1 + x(t),\ x(0) = 0.$$

a) Find $x_3(t)$ and the exact solution $\hat{x}(t)$ (notice that the equation is separable). Integrate $x'(s)$ from 0 to t, we get

$$x(t) = x(t) - x(0) = \int_0^t x'(s)\,ds = \int_0^t 1 + x(s)\,ds,$$

so the Picard iteration is given by

$$x_{n+1}(t) = \int_0^t 1 + x_n(s)\,ds,\ \ x_0 \equiv 0.$$

Compute inductively

$$x_1(t) = t,\ \ x_2(t) = t + \frac{t^2}{2},\ \ x_3(t) = t + \frac{t^2}{2} + \frac{t^3}{6}.$$

The exact solution is given by $\hat{x}(t) = e^t - 1$.

b) Show that the terms involving t, t^2, t^3 in $x_3(t)$ are the same as those of the Taylor series of the exact solution.

This follows from that

$$\hat{x}(t) = e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots + \frac{t^n}{n!} + \cdots .$$

c) Explain why we have $\lim_{n\to\infty} x_n(t) = \hat{x}(t)$. One may look at the map

$$T : x \to Tx, \qquad (Tx)(t) := \int_0^t 1 + x(s)\, ds.$$

Observe that

$$|(Tx)(t) - (Ty)(t)| = |\int_0^t x(s) - y(s)\, ds| \leq |t| \sup_{|s|\leq|t|} |x(s) - y(s)|.$$

Hence if $x \in C[-L, L]$ fo some $0 < L < 1$, then

$$||Tx - Ty|| := \sup_{|t|<L} |(Tx)(t) - (Tx)(t)| \leq L||x - y||.$$

So T is a contraction mapping from $(C[-L, L], ||\cdot||)$ to self, so by the Banach fixed point theorem we know that the Picard iteration above at least converges for $|t| < 1$.

Problem 5

Let M be a closed subspace of a Hilbert space X, which by the projection theorem is given by the direct sum $X = M \oplus M^\perp$, where $M^\perp$ denotes the orthogonal complement.

a) Show that $M^\perp$ is closed. Assume that $x_n \to x$ with $x_n \in M^\perp$, then we need to show that $x \in M^\perp$, i.e. $\langle x, m\rangle = 0$ for all $m \in M$. But notice that all

$$\langle x_n, m\rangle = 0$$

thus

$$|\langle x, m\rangle| = |\langle x, m\rangle - \langle x_n, m\rangle| \leq ||x - x_n|| \cdot ||m|| \to 0$$

gives $\langle x, m\rangle = 0$.

b) Show that the orthogonal projection onto M is self-adjoint. Denote by P the projection mapping, we have

$$\langle Px, y\rangle = \langle Px, y - Py\rangle + \langle Px, Py\rangle = \langle Px, Py\rangle$$

since $y - Py \in M^\perp \perp Px \in M$, the same argument gives

$$\langle x, Py\rangle = \langle Px, Py\rangle.$$

Hence

$$\langle Px, y\rangle = \langle x, Py\rangle$$

and P is self-adjoint.

Problem 6

Let $L^2[0,\pi]$ be equipped with the inner product

$$\langle f, g\rangle = \int_0^\pi f(t)\overline{g(t)}\, dt.$$

Apply Gram-Schmidt's algorithm to the basis

$$\{\sin t, \sin(2t), \cdots, \sin(nt), \cdots\}$$

of $L^2[0,\pi]$ to obtain an orthonormal basis. One can use $2\sin a \sin b = \cos(a-b) - \cos(a+b)$ to compute

$$\langle \sin nt, \sin mt\rangle = 0, \quad n \neq m.$$

and

$$\langle \sin nt, \sin nt\rangle = \frac{\pi}{2}.$$

Thus the orthonormal basis is given by $\{\sqrt{\frac{2}{\pi}}\sin nt\}_{n\geq 1}$.