Exercise set 6

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1 Recall that in the class we have proved the following Legendre transform formula

$$\sup_{x>0}\left\{xy-\frac{x^p}{p}\right\}=\frac{y^q}{q},\ y>0,\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1.$$

Use the above formula to show that

$$\sup_{x\in\mathbb{R}^n}\left\{x\cdot y-\frac{||x||_p^p}{p}\right\}=\frac{||y||_q^q}{q},\ y\in\mathbb{R}^n,\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

where $||\cdot||_p$ denotes the p-norm on $\mathbb{R}^n$.

2 Let $\phi(t,x)$ be a smooth function on $[0,1]\times[0,1]$. Assume that ϕ is convex in t (that is to say, $\phi_{tt}(t,x)\geq 0$ for all $0\leq t,x\leq 1$). Prove that

$$\tilde{\phi}(t)=\log\int_0^1 e^{\phi(t,x)}\,dx$$

is convex in $0\leq t\leq 1$ by computing the second order derivative of $\tilde{\phi}$ (from the class we know that this convexity property can be seen as a generalized Hölder inequality).

3 Let $\phi(t,x)$ be a smooth function on $(0,1)\times(0,1)$. Assume that ϕ is convex in t. Show that in general

$$\hat{\phi}(t)=-\log\int_0^1 e^{-\phi(t,x)}\,dx$$

is not convex in $0<t<1$ (try the example $\phi(t,x)=t\log x$). (This implies that the naive version of the opposite Hölder inequality is in general not true, the right version is called the Prekopa theorem, which says that $\hat{\phi}$ is convex if ϕ is convex in both t and x, you do not need to prove this).

4 Consider the following differential operator $D: C^1(\mathbb{R})\to C(\mathbb{R})$ defined by

$$Df=f'-3f,$$

find the kernel and image of D.