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Sciences

$\boxed{1}$ Show that $U \in \mathcal{M}_{n \times n}(\mathbb{C})$ is unitary if and only if its column vectors define an orthonormal basis of $\mathbb{C}^n$.

$\boxed{2}$ Suppose that A and B are unitarily equivalent, meaning that there exists a unitary matrix U such that

$$B = U^* A U.$$

Prove that A is positive definite (semi-definite) if and only if B is positive definite (semi-definite).

$\boxed{3}$ Find the singular value decomposition for the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

$\boxed{4}$ (Hard exercise) Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$. Show that AA^* and A^*A have the same positive eigenvalues. Moreover, the λ-eigenspaces ($\lambda > 0$) of AA^* and A^*A have the same dimension over $\mathbb{C}$.