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Exercise set 2

1 Let X be a vector space of dimension n and let $\mathcal{L}(x)$ denote the vector space of all linear mappings on X. Show that $\mathcal{L}(x)$ is isomorphic to $\mathcal{M}_n$, the space of all $n \times n$ matrices.

2 Let $\mathcal{P}_n$ be the space of polynomials of degree at most n. We pick as basis $\mathcal{B} = \{p_0(x), ..., p_n(x)\}$ where $p_0(x) = 1, p_1(x) = x, p_2(x) = x(x-1), ..., p_j(x) = x(x-1)\cdots(x-j+1), ...p_n(x) = x(x-1)\cdots(x-n+1)$. We consider the difference operator D on $\mathcal{P}_n$ defined by $Dp(x) = p(x+1) - p(x)$.

a) Find the matrix representations of D with respect to the mononmial basis $\mathcal{M} = \{1, x, .., x^n\}$ and the basis $\mathcal{B} = \{p_0(x), ..., p_n(x)\}$, respectively. (You might for the sake of simplicity restrict your discussion to $n = 3$.)

b) Show that $Dp_j = jp_{j-1}$ for $j = 1, ..., n$.

c) Use a) to conclude that for $p \in \mathcal{P}_n$ we have $D^n p \neq 0$ and $D^{n+1}p = 0$.

3 Let A be the matrix $A = \begin{pmatrix} 15 & -6 & 2 \\ 35 & -14 & 5 \\ 7 & -3 & 2 \end{pmatrix}$. Find the eigenvalues and generalized eigenspaces of A.

4 Let X be a finite-dimensional vector space and $T : X \to X$ a linear mapping. Show that

$$X \supseteq \mathrm{im}(T) \supseteq \mathrm{im}(T^2) \supseteq \cdots$$

and that there exists an integer k such that $\mathrm{im}(T^k) = \mathrm{im}(T^{k+1})$.