Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods Fall 2022

Solutions to exercise set 8

Below follows *one* possible solution to the exercise set.

1 We consider the Newton iteration

$$T(x) := x - \frac{f(x)}{f'(x)} = \frac{1}{2}\left(x + \frac{3}{x}\right),$$

and want to show that T maps $[\sqrt{3}, \infty)$ to itself. Moreover, we want to use the Banach fixed point theorem to show that

$$\lim_{n\to\infty} T^n(x) = \sqrt{3}$$

for every $x \geq \sqrt{3}$.

To show that T maps $[\sqrt{3}, \infty)$ to itself, we note that for $x \geq \sqrt{3}$,

$$T'(x) = \frac{1}{2}\left(1 - \frac{3}{x^2}\right) \geq 0,$$

as $x^2 \geq 3$ for all $x \in \mathbb{R}$. This shows that $T(x)$ is a monotonically increasing function of x for $x \in [\sqrt{3}, \infty)$. Thus the lowest value of the image of T is given by

$$T(\sqrt{3}) = \frac{1}{2}\left(\sqrt{3} + \frac{3}{\sqrt{3}}\right) = \sqrt{3}. \tag{1}$$

In particular, $T([\sqrt{3}, \infty)) = [\sqrt{3}, \infty)$, so T maps $[\sqrt{3}, \infty)$ to itself.

We have already seen in (1) that $\sqrt{3}$ is a fix point of T. We would like to use Banach fixed point theorem to show that $\sqrt{3}$ is the unique fixed point, and that no matter which starting point it will converge to $\sqrt{3}$.

To use Banach fixed point theorem, we need to show that $[\sqrt{3}, \infty)$ is complete, and that T is a contraction. For completeness, it is enough to prove that $[\sqrt{3}, \infty)$ is closed, as any closed subset of a complete space is complete. However, this follows from the definition as the complement $(-\infty, \sqrt{3})$ is open, so $[\sqrt{3}, \infty)$ is closed. Hence, we can conclude that $([\sqrt{3}, \infty), |\cdot|)$ is a complete metric space.

To show that T is a contraction, we estimate

$$\begin{aligned} |T(x) - T(y)| = \frac{1}{2}\left|x - y + \frac{3}{x} - \frac{3}{y}\right| &= \frac{1}{2}\left|x\left(1 - \frac{3}{xy}\right) - y\left(1 - \frac{3}{xy}\right)\right| \\ &= \frac{1}{2}|x - y|\left|1 - \frac{3}{xy}\right| \\ &\leq \frac{1}{2}|x - y|. \end{aligned}$$

For the last inequality, we used that,

$$0 \leq 1 - \frac{3}{xy} \leq 1, \quad x, y \in [\sqrt{3}, \infty).$$

We can therefore conclude that T is a contraction with contraction constant

$$L = 1/2 < 1.$$

By Banach fixed point theorem there exists a unique fixed point x^* such that

$$\lim_{n \to \infty} T^n(x) = x^*,$$

for all $x \in [\sqrt{3}, \infty)$. By (1) we can conclude that $x^* = \sqrt{3}$. The fixed point can also be found by solving,

$$x = T(x) = \frac{1}{2}\left(x + \frac{3}{x}\right) \quad \Longrightarrow \quad x = \frac{3}{x},$$

which has the solution $x = \sqrt{3}$ in the interval $[\sqrt{3}, \infty)$.

2 Let $G : (C[0,1], ||\cdot||_\infty) \to (C[0,1], ||\cdot||_\infty)$ be defined by

$$(Gx)(t) = \int_0^t sx(s)\, ds, \ 0 \leq t \leq 1.$$

We want to show that G is a contraction with zero function as the unique fixed point. Given any $x, y \in C[0,1]$ and $0 \leq t \leq 1$, then

$$\begin{aligned} |Gx(t) - Gy(t)| = \left| \int_0^t s(x(s) - y(s))ds \right| \leq & \int_0^t s|x(s) - y(s)|ds \\ \leq & \|x - y\|_\infty \int_0^t sds \\ = & \frac{1}{2}\|x - y\|_\infty t^2 \leq \frac{1}{2}\|x - y\|_\infty. \end{aligned}$$

Taking the supremum over all $0 \leq t \leq 1$ gives

$$\|Gx - Gy\|_\infty \leq \frac{1}{2}\|x - y\|_\infty,$$

which shows that G is a contraction, on the complete space $(C[0,1], \|\cdot\|_\infty)$. By Banach fixed point theorem, there exists a unique fixed point x^* such that $Gx^* = x^*$. Moreover, the fixed point is given by,

$$x^*(t) = \lim_{n \to \infty} G^n x(t),$$

for any $x \in C[0,1]$. From the definition of G, we see that

$$\|Gx\|_\infty \leq \frac{1}{2}\|x\|_\infty.$$

Iterating this inequality yields,

$$\|G^n x\|_\infty \le \left(\frac{1}{2}\right)^n \|x\|_\infty \xrightarrow{n\to\infty} 0.$$

Thus, $G^n x \to 0$ as n goes to ∞ for any $x \in C[0,1]$. Indeed, by direct computation, we see that

$$G0(t) = \int_0^t 0ds = 0,$$

which verifies that 0 is the unique fixed point.

3 Apply Picard iteration to

$$x'(t) = 1 + x^2, \; x(0) = 0.$$

Find x_3 and the exact solution (notice that the equation is separable), and show that the terms involving $t, t^2, \cdots, t^5$ in $x_3(t)$ are the same as those of the Taylor series of the exact solution.

Let us first solve the initial-value problem analytically (exact). Using that the differential equation we want to solve is separable we find

$$\frac{dx}{1+x^2} = dt$$

which gives after integrating,

$$\arctan(x) = t + C,$$

and thus

$$x = x(t) = \tan(t + C).$$

Since $x(0) = 0$ we find that $0 = \tan(C)$, so we may take $C = 0$. Thus the solution is

$$x(t) = \tan(t).$$

Next, we find an approximation for the solution of the initial-value problem using Picard iteration. We want x_3 where

$$x_0(t) = x_0, \qquad x_{n+1}(t) = x_0 + \int_{t_0}^t f(s, x_n(s))\, ds, \qquad n = 0, 1, 2, \dots$$

We find with $x_0 = t_0 = 0$,

$$\begin{aligned}
x_1(t) &= \int_0^t f(s,0)\, ds = \int_0^t 1\, ds = t \\
x_2(t) &= \int_0^t f(s,s)\, ds = \int_0^t 1 + s^2\, ds = t + \frac{1}{3}t^3 \\
x_3(t) &= \int_0^t f(s,(s+s^3/3))\, ds = \int_0^t 1 + \left(s + \frac{s^3}{3}\right)^2\, ds \\
&= t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63}.
\end{aligned}$$

On the other hand, the Taylor series of $t \mapsto \tan(t)$ at $t = 0$ is given by

$$\tan(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + O(t^7).$$

Comparing we see that the terms up to order 5 of $x_3(t)$ and the Taylor series of analytic solution $\tan(t)$ agree.