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TMA4145
Linear Methods
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Solutions to exercise set 4

Below follows *one* possible solution to the exercise set.

1 We want to show that $U \in \mathcal{M}_{n\times n}(\mathbb{C})$ is unitary if and only if the column vectors define an orthonormal basis of $\mathbb{C}^n$.

Recall that for two $n \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$ the matrix product is defined as the matrix AB with the elements

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk}.$$

Assume that $U = (u_{ij})$ is a unitary matrix, and let $u_j = (u_{1,j}, u_{2,j}, \dots, u_{n,j})^T$ denote the j column of the matrix $U = (u_{ij})$. Since U is unitary we know that $U^*U = I_n$, where $U^* = (\bar{u}_{ji})$ and I_n is the identity on $\mathbb{C}^n$. Thus we have

$$(U^*U)_{ik} = \sum_{j=1}^{n} \bar{u}_{ji}u_{jk} = \delta_{ik} := \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

However, the sum is nothing else than the inner product of the columns of U. Namely,

$$\langle u_k, u_i \rangle = \sum_{j=1}^{n} \bar{u}_{ji}u_{ki} = \delta_{ik}.$$

This shows that the columns of a unitary matrix are orthonormal.

On the other hand, given a set of orthonormal vectors $\{u_j = (u_{1,j}, u_{2,j}, \dots, u_{n,j})^T\}_{j=1}^{n}$, then the matrix U is unitary, as

$$(U^*U)_{ik} = \sum_{j=1}^{n} \bar{u}_{ji}u_{jk} = \langle u_k, u_i \rangle = \delta_{ik},$$

and

$$(UU^*)_{ik} = \sum_{j=1}^{n} u_{ji}\bar{u}_{jk} = \langle u_i, u_k \rangle = \delta_{ki}.$$

This shows that $U^*U = I_n = UU^*$, and so U is unitary.

2 Assume that A and B are unitary equivalent matrices. That is, there exists some unitary matrix U such that $B = U^*AU$. Show that A is positive definite if and only if B is positive definite.

A matrix A is called positive definite if $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{C}^n \backslash \{0\}$. So assume that A is positive definite, then for any $x \in \mathbb{C}^n \backslash \{0\}$ we have that

$$\langle Bx, x\rangle = \langle U^*AUx, x\rangle = \langle AUx, Ux\rangle = \langle A(Ux), Ux\rangle > 0,$$

as A is positive definite.

On the other hand, assume that B is positive definite. Then, for any $x \in \mathbb{C}^n \backslash \{0\}$ we have

$$0 < \langle B(U^*x), U^*x\rangle = \langle UBU^*x, x\rangle = \langle Ax, x\rangle,$$

as $A = UBU^*$. This shows that A is positive definite. The argument for semi-positiveness is the same

3 Let us compute the singular values of A. Recall that these are the square roots of the non-zero eigenvalues of the self-adjoint matrix

$$AA^* = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \quad \text{or} \quad A^*A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

In the first case we get a 2×2-matrix and in the second case we get a 3×3-matrix, so for simplicity we use AA^* for the computation. The eigenvalues of AA^* are 6 and 0 (for those, who have decided to use A^*A, the eigenvalues are $6, 0, 0$). Hence $\sigma_1 = \sqrt{6}$ is the only singular value of A, which fits well with the fact that A has rank one. Consequently, Σ is given by

$$\Sigma = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us look at the eigenvectors of A^*A. A bit of computation yields

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \; v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The vectors v_1, v_2, v_3 form an orthonormal basis for $\mathbb{R}^3$, and serve as the columns of V:

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Finally we find the columns of U. The first column is given by

$$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since we only had one singular value σ_1, the second column of U is obtained by completing $\{u_1\}$ to an orthonormal basis for $\mathbb{R}^2$. This is achieved by choosing u_2 orthogonal to u_1, and by inspection we see that we can choose $u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Hence

$$A = U\Sigma V^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

4 For $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ we want to show that AA^* and A^*A have the same positive eigenvalues. Moreover, the λ-eigenspaces of AA^* and A^*A have the same dimension over $\mathbb{C}$.

First, note that any eigenvalues of A^*A and AA^* are necessarily positive. Let λ be an eigenvalue of A^*A with the corresponding eigenvector v. Then

$$\lambda\langle v, v\rangle = \langle A^*Av, v\rangle = \langle Av, Av\rangle \geq 0,$$

and similarly for AA^*.

Now, let $\lambda > 0$ be a positive eigenvalue of A^*A, with the corresponding eigenvector v. Then, we claim that Av is an eigenvector of AA^* with eigenvalue λ.

$$AA^*(Av) = A(A^*A)v = A(\lambda v) = \lambda Av.$$

Note that Av is necessarily not 0, as v is an eigenvalue of A^*A. If Av was zero, then $A^*Av = A^*0 = 0$, which contradicts the fact that $A^*Av = \lambda v$ for $\lambda > 0$. This shows that any eigenvalue of A^*A is also an eigenvalue of AA^*.

Similarly, if λ is an eigenvalue of AA^* with eigenvector u, then A^*u is an eigenvalue of A^*A with eigenvalue λ. Namely,

$$A^*A(A^*v) = A^*(AA^*)v = A^*(\lambda v) = \lambda A^*v.$$

This shows that A^*A and AA^* has the same eigenvalues.

Now, assume that for an eigenvalue $\lambda > 0$ the λ-eigenspace of A^*A is k-dimensional. Then there exists k linearly independent vectors $v_1, \ldots v_k$ such that

$$c_1v_1 + \ldots + c_kv_k = 0, \quad \text{if and only if} \quad c_j = 0, \text{ for } 1 \leq j \leq k.$$

From here, consider the vectors $Av_1, \ldots, Av_k$ which lies in the λ-eigenspace of AA^*, and assume that

$$0 = \sum_{j=1}^{k} c_jAv_j$$

for some choice of c_j. Then, as A and A^* are linear transformation, it follows that

$$0 = A^*(0) = A^*\left(\sum_{j=1}^{k} c_jAv_j\right) = \sum_{j=1}^{k} c_jA^*Av_j = \lambda\sum_{j=1}^{k} c_jv_j.$$

From the linearly independence of v_j we can conclude that the eigenvectors $Av_1, \ldots Av_k$ must be linearly independent. Thus, the dimension of the λ-eigenspace of AA^* must be at least the same as that of A^*A. Namely,

$$\dim(\lambda\text{-eigenspace of } AA^*) \geq \dim(\lambda\text{-eigenspace of } A^*A).$$

However, by running the same argument again, but starting with the λ-eigenspace of AA^* we get that

$$\dim(\lambda\text{-eigenspace of } AA^*) \leq \dim(\lambda\text{-eigenspace of } A^*A),$$

as A^*v_j is a λ eigenvector of A^*A for each λ-eigenvector v_j of AA^*. This implies that the dimension is the same.

Note that this also implies that the dimension of the 0-eigenspace is the same for both matrices. Since the λ-eigenspaces only intersect trivially for different eigenvalues $\lambda > 0$, it follows that the sum of all dimension of the eigenspaces gives the full dimension n. Thus

$$\begin{aligned}\dim(0\text{-eigenspace of } AA^*) =& n - \sum_{\lambda>0} \dim(\lambda\text{-eigenspace of } AA^*)\\ =& n - \sum_{\lambda>0} \dim(\lambda\text{-eigenspace of } A^*A)\\ =& \dim(0\text{-eigenspace of } A^*A).\end{aligned}$$