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TMA4145
Linear Methods
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Solutions to exercise set 3

Below follows *one* possible solution to the exercise set.

1 The matrix $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is upper-triangular, and so the eigenvalues are given by the diagonal elements. Namely, $\lambda_1 = 1$ and $\lambda_2 = 2$. Since A has two distinct eigenvalues it is diaognalizable.

A real matrix is called normal if $AA^T = A^T A$. By direct computation we have

$$AA^T = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 4 \end{pmatrix},$$
$$A^T A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 13 \end{pmatrix}.$$

This shows that $A^T A \neq AA^T$, and so A is not normal.

2 Consider the matrix

$$A = \begin{pmatrix} -1.8 & 0 & -1.4 \\ -5.6 & 1 & -2.8 \\ 2.8 & 0 & 2.4 \end{pmatrix}.$$

We want to diagonalize it, and find A^m for $m \in \mathbb{N}$. Let us therefore start by finding the eigenvalues of A, by solving

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 1)(\lambda + 0.4) = 0,$$

in order to see that the eigenvalues are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = -0.4$. For the eigenvectors, we find that

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Let V and Σ be the matrices

$$V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5} \end{pmatrix}.$$

then the inverse matrix is given by

$$V^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix},$$

and so the diagonalization of A is given by

$$A = V\Sigma V^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix}.$$

To calculate A^m, for $m \in \mathbb{N}$, we note that

$$A^2 = V\Sigma V^{-1} V\Sigma V^{-1} = V\Sigma^2 V^{-1}.$$

Thus, a simple induction argument shows that

$$A^m = V\Sigma^m V^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-\frac{2}{5})^m \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix}.$$

Moreover, the entry-wise limit $\lim_{m\to\infty} A^m$ does exists, and is given by

$$\lim_{m\to\infty} A^m = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -2 \\ 2 & 0 & 2 \end{pmatrix}.$$

3 Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Let us define $A^* \in \mathcal{M}_{n\times m}(\mathbb{C})$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle$$

for all column vectors $x \in \mathbb{C}^n, y \in \mathbb{C}^m$, where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian inner product. We want to show that $A^* = \bar{A}^T$.

Recall that the standard Hermitian inner product on $\mathbb{C}^n$ is defined by

$$\langle x, y\rangle = \overline{y}^T x,$$

for all $x, y \in \mathbb{C}^n$, and similarly on $\mathbb{C}^m$. For $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ we have

$$\langle x, A^* y\rangle = \langle Ax, y\rangle = \overline{y}^T Ax = (A^T\overline{y})^T x = (\overline{\bar{A}^T y})^T x = \langle x, \bar{A}^T y\rangle.$$

This shows that $A^* = \bar{A}^T$.

4 Let $A \in \mathcal{M}_n(\mathbb{C})$ be a self-adjoint matrix, that is $A = A^*$. We want to show that eigenvectors corresponding to different eigenvalues are orthogonal.

We claim that any eigenvalue of A is necessarily real. Let λ be an eigenvalue of A with the corresponding eigenvector v. Then

$$\lambda\langle v, v\rangle = \langle Av, v\rangle = \langle v, A^* v\rangle = \langle v, Av\rangle = \bar{\lambda}\langle v, v\rangle.$$

Thus $\lambda = \bar{\lambda}$, which means that the eigenvalue is real.

Consider any two distinct eigenvalues $\lambda \neq \mu$ with the corresponding eigenvectors v and u, respectively. Then

$$\lambda\langle v, u\rangle = \langle Av, u\rangle = \langle v, Au\rangle = \bar{\mu}\langle v, u\rangle = \mu\langle v, u\rangle.$$

The last inequality follows from the fact that the eigenvalues are real. Since $\lambda \neq \mu$, the equality can only hold if $\langle v, u\rangle = 0$, which means that the eigenvectors v and u are orthogonal.