Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods Fall 2022

Solutions to exercise set 2

Below follows *one* possible solution to the exercise set.

1 Let X be a vector space of dimension n, and $\mathcal{L}(X)$ be the spaces of linear mappings on X. We want to show that $\mathcal{L}(X)$ is isomorphic to $\mathcal{M}_n$, the space of $n \times n$ matrices.

Let $\mathcal{B} = \{e_j\}_{j=1}^n$ be a basis on X. From here we can define a linear map $\varphi : X \to \mathbb{F}^n$ by

$$\varphi\left(\sum_{i=1}^n \alpha_i e_i\right) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

In particular, φ maps the basis of X to the standard basis on $\mathbb{F}^n$. The linearity of φ follows from the linearity of vector addition on $\mathbb{F}^n$. Note that φ is an isomorphism from X to $\mathbb{F}^n$. To see this, note that if $\varphi(x) = \varphi(y)$, where

$$x = \sum_{i=1}^n x_i e_i, \quad y = \sum_{i=1}^n y_i.$$

Then $x_i = y_i$ for all i, and thus $x = y$, and φ is injective. Moreover, for each $\xi \in \mathbb{F}^n$, where

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix},$$

we can define the vector $v \in X$ given by

$$v = \sum_{i=1}^n \xi_i e_i.$$

Then $\varphi(v) = \xi$, and so φ is also surjective. This shows that φ is an isomorphism, and moreover it has a well-defined inverse given by

$$\varphi\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \sum_{i=1}^n \alpha_i e_i.$$

Note that any linear transformation on X is uniquely determined by how it acts on the basis vectors. Namely, for any $T \in \mathcal{L}(X)$, and any $x \in X$, we have

$$T(x) = T\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{i=1}^n \alpha_i T(e_i).$$

We therefore define the map from $\Phi : \mathcal{L}(X) \to \mathcal{M}_n$ given by

$$T \mapsto \Phi(T) = M_T = \begin{pmatrix} \varphi(T(e_1)) & \dots & \varphi(T(e_n)) \end{pmatrix}.$$

That is, the j column of the matrix M_T is given by the vector $T(e_j)$.

We claim that this map is a bijection, and hence an isomorphism.

First of, for any matrix $M \in \mathcal{M}_n$ we can define the linear transformation T_M by defining it on each of the basis vectors given by

$$T(e_j) = \varphi^{-1}(M_j),$$

where M_j is the j^{th} column of the matrix M. Then

$$\varphi(T_M) = \begin{pmatrix} M_1 & \dots & M_n \end{pmatrix} = M,$$

which shows that the map Φ is surjective. Moreover, assume that for some $S, T \in \mathcal{L}(X)$ we have

$$\Phi(S) = \Phi(T).$$

Then we have

$$\varphi(S(e_j)) = \varphi(T(e_j)),$$

for each j. Since we showed that φ is injective it follows that $S(e_j) = T(e_j)$. Moreover, as S and T coincide on each basis vector they have to be the same. That is, $S = T$ and so Φ is injective. This shows that Φ is an isomorphism.

2 Let $p_0 = 1$, and define the polynomials p_j for $1 \leq j \leq n$ by

$$p_j(x) = \prod_{i=1}^{j} (x - i + 1).$$

Moreover, define the map $Dp(x) = p(x+1) - p(x)$

a) Find the matrix representation of D with respect to the basis $\mathcal{B} = \{p_0, \dots, p_n\}$ and the basis $\mathcal{M} = \{1, x, \dots x^n\}$.

Let us start with the basis $\mathcal{M}$. Note that $D1 = 0$. By the binomial formula, we know that for $j > 1$

$$(x+1)^j = \sum_{i=0}^{j} \binom{j}{i} x^{j-i}.$$

From here we have

$$Dx^j = (x+1)^j - x^j = (x+1)^j = \sum_{i=1}^{j} \binom{j}{i} x^{j-i} = \sum_{i=0}^{j-1} \binom{j}{j-i} x^i.$$

Thus the matrix components of $[D]_{\mathcal{M}}$ is given by

$$D_{ij} = \begin{cases} \binom{j-1}{j-i} & j > i, \\ 0 & j \leq i. \end{cases}$$

For $n = 3$, the matrix looks like

$$[D]_{\mathcal{M}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the basis $\mathcal{B}$ we first note a few nice properties. Firstly, note that

$$p_j(x) = (x - j + 1)p_{j-1}(x).$$

Secondly, we note that

$$\begin{aligned} p_j(x+1) = \prod_{i=1}^{j}((x+1) - i + 1) = &\prod_{i=1}^{j}(x - (j-1) + 1) \\ =&(x+1)\prod_{i=1}^{j-1}(x - j + 1) \\ =&(x+1)p_{j-1}(x). \end{aligned}$$

Thus, the difference operator D is given by

$$Dp_j(x) = p_j(x+1) - p_j(x) = (x+1)p_{j-1}(x) - (x-j+1)p_{j-1}(x) = jp_{j-1}(x),$$

for $j \geq 1$. We still have $D1 = 0$. This gives the matrix components

$$[D_{ij}]_{\mathcal{B}} = \begin{cases} j-1 & i = j-1 \\ 0, & \text{else.} \end{cases}$$

For $n = 3$, the matrix looks like

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) It was shown in **a**) that $Dp_j(x) = jp_{j-1}(x)$.

c) It is enough to consider the action on the basis elements as D is a linear transformation. You should convince yourself about this fact. We choose to work with the basis $\mathcal{B}$.
Since we have $Dp_j(x) = jp_{j-1}(x)$, it follows that for $1 \leq i \leq j$

$$D^i p_j(x) = \frac{j!}{(j-i)!}p_{j-i}(x).$$

In particular, $D^j p_j(x) = j!p_0(x) = j!$, and so $D^{j+1}p_j(x) = 0$ for any $1 \leq j \leq n$. but this means that for any $p \in \mathcal{P}_n$, we have

$$\begin{aligned} D^n p(x) =&D^n\left(\sum_{i=0}^{n}\alpha_i p_i(x)\right) = n!\alpha_n p_0(x) = n!\alpha_n, \\ D^{n+1} =&n!\alpha_n Dp_0(x) = 0. \end{aligned}$$

This proves the claim. Note that the way the problem is stated is not completely correct as there exists polynomials $p \in \mathcal{P}_n$ such that $D^n p(x) = 0$. Namely all the polynomials of degree strictly less than n, that is $\alpha_n = 0$. However, D^n does not vanish on $\mathcal{P}_n$, while D^{n+1} does.
For $n = 3$ we have.

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [D]^2_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$[D]^3_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [D]^4_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

3 Let consider the matrix

$$A = \begin{pmatrix} 15 & -6 & 2 \\ 35 & -14 & 5 \\ 7 & -3 & 2 \end{pmatrix}.$$

Find the eigenvalues and the generalized eigenspaces.

We start by finding the eigenvalues.

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = (1 - \lambda)^3.$$

The matrix has eigenvalues $\lambda_i = 1$ for $i \in \{1, 2, 3\}$. From here we want to find the eigenvectors.

$$A - I = \begin{pmatrix} 14 & -6 & 2 \\ 35 & -15 & 5 \\ 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 7 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can therefore conclude that the eigenvectors are given by

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}.$$

The generalized eigenspace is defined as

$$E_\lambda = \{v \in \mathbb{R}^3 : \text{There exists } m \in \mathbb{N} \text{ such that } (A - \lambda I)^m v = 0, \, (A - \lambda I)^{m-1} v \neq 0\}.$$

To find a generalized eigenvector, we want to see if we can find a vector z such that

$$(A - I)z \neq 0, \quad (A - I)^2 z = 0.$$

Note that $(A - I)^2 = 0$. In particular we can choose

$$z = v_1 \times v_2 = \begin{pmatrix} -7 \\ 3 \\ -1 \end{pmatrix}.$$

Then

$$(A-I)z = \begin{pmatrix} 14 & -6 & 2 \\ 35 & -15 & 5 \\ 7 & -3 & 1 \end{pmatrix} \begin{pmatrix} -7 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -118 \\ -295 \\ -59 \end{pmatrix} \neq 0$$

while $(A-I)^2 z = 0$. We therefore have the generalized eigenspace for the eigenvalue $\lambda = 1$ given by

$$E_1 = \text{span}\{v_1, v_2, z\} = \mathbb{R}^3.$$

4 Let X be a finite dimensional vector space, and $T : X \to X$ be a linear transformation. Prove that

$$X \supseteq \text{im}(T) \supseteq \text{im}(T^2) \supseteq \dots,$$

and that there exist an integer k such that $\text{im}(T^k) = \text{im}(T^{k+1})$.

Recall that the image is defined by

$$\text{im}(T) = \{x \in X : x = T(y) \text{ for some } y \in X\}.$$

We can prove the first part by a simple induction argument. The base case it follows from the definition of the image of T. That is $\text{im}(T) \subseteq X$.

Assume that we have $X \supseteq \text{im}(T) \supseteq \ldots \supseteq \text{im}(T^{j-1})$. Then by definition of the image of T^j we know that if $x \in \text{im}(T^j)$ then there exists some $y \in X$ such that

$$x = T^j(y) = T^{j-1}(Ty) = T^{j-1}(z) \in \text{im}(T^{j-1}).$$

For the second part we note that the image of T^j is a subspace. This follows from the fact that T is a linear transformation. By the first part we have for each integer j

$$\text{im}(T^{j-1}) \supseteq \text{im}(T^j).$$

In particular, this implies that $\dim(\text{im}(T^j)) \leq \dim(\text{im}(T^{j-1}))$.

Let $n = \dim(X)$. Then for every integer $0 \leq k \leq n$ we have that either $\dim(\text{im}(T^k)) = \dim(\text{im}(T^{k-1}))$ or $\dim(\text{im}(T^k)) < \dim(\text{im}(T^{k-1}))$. Consider first the case that for some integer k we have $\dim(\text{im}(T^k)) = \dim(\text{im}(T^{k-1})) = m$. Then there exists m linear independent vectors $v_1, \ldots v_m \in \text{im}(T^k)$ such that $\text{span}\{v_1, \ldots v_m\} = \text{im}(T^k)$. Moreover, by the first part $\text{im}(T^k) \subseteq \text{im}(T^{k-1})$, and so $v_1, \ldots v_m$ are m linearly independent vectors in $\text{im}(T^{k-1})$. Since $\dim(\text{im}(T^{k-1})) = m$, it follows from lemma 1.6 in the notes, $\{v_1, \ldots v_m\}$ is a basis for $\text{im}(T^{k-1})$. Thus for any $x \in \text{im}(T^{k-1})$ we have

$$x = \sum_{i=1}^{m} \alpha_i v_i \in \text{span}\{v_1, \ldots, v_m\} = \text{im}(T^k).$$

This shows that $\text{im}(T^{k-1}) \subseteq \text{im}(T^k)$, and thus we have equality.

Assume on the other hand that $\dim(\text{im}(T^k)) < \dim(\text{im}(T^{k-1}))$. Since X is finite dimensional, there exists an integer $k \leq n$ such that $\dim(\text{im}(T^k)) = 0$. Otherwise we would have had an equality of the dimensions. Since $\text{im}(T^k)$ is a vector space, and the only zero dimensional vector space is $\{0\}$, it follows that $\text{im}(T^k) = \{0\}$. However, as T is a linear transformation, we know that $T(0) = 0$. Thus, for any $x \in \text{im}(T^{k+1})$

$$x = T^{k+1}(z) = T(T^k(z)) = T(0) = 0.$$

This shows that

$$\text{im}(T^{k+1}) = \{0\} = \text{im}(T^k).$$