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Solutions to exercise set 12

Below follows *one* possible solution to the exercise set.

1 Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard basis in the sequence space ℓ^∞. Show that the series $\sum_{n=1}^{\infty} a_n e_n$ converges in ℓ^∞ if and only if $\{a_n\}_{n\in\mathbb{N}}$ converges to zero.

We start by assuming that $a_n \to 0$. To show that $\sum_{n=1}^{\infty} a_n e_n$ converges in ℓ^∞, we will assume that the partial sums $s_N = \sum_{n=1}^{N} a_n e_n$ converges to a. Clearly,

$$a - s_N = (0, \dots, 0, a_{N+1}, a_{N+2}, \dots),$$

and thus we note that,

$$\|a - s_N\|_\infty = \sup_{l\geq N+1} |a_l|.$$

By the assumption that $a_N \to 0$ as $N \to \infty$, it follows that

$$\|a - s_N\|_\infty = \sup_{l\geq N+1} |a_l| \xrightarrow{N\to\infty} 0,$$

and so the series converges in ℓ^∞.

Conversely, assume that $\sum_{n=1}^{\infty} a_n e_n$ converges in ℓ^∞. This means that the partial sums $s_N = \sum_{n=1}^{N} a_n e_n$ converge to some element $x = (x_1, x_2, \dots) \in \ell^\infty$. It necessarily follows that $x_i = a_i$ for each $i \in \mathbb{N}$, since for any $N > i$,

$$x - s_N = (x_1 - a_1, \dots, x_i - a_i, \dots, x_N - a_N, x_{N+1}, x_{N+2}, \dots).$$

In particular,

$$|x_i - a_i| \leq \sup_{j\in\mathbb{N}} |x_j - (s_N)_j| = \|x - s_N\|_\infty \xrightarrow{N\to\infty} 0.$$

By the assumption that $s_N \to x$ in ℓ^∞, it follows that $s_N \to a = (a_1, a_2, \dots)$ by the fact that $x_i = a_i$ for each i. However, this implies that

$$|a_{N+1}| \leq \sup_{l\geq N+1} |a_i| = \|a - s_N\|_\infty \xrightarrow{N\to\infty} 0,$$

which shows that $a_n \to 0$ as $n \to \infty$.

2 Show that if a normed space $(X, ||\cdot||)$ has a Schauder basis, then it is separable.

Recall that X is separable if there exists a countable dense subset of X. We claim that such a subset is given all finite linear combinations of basis elements with rational coefficients,

$$Y = \left\{ q \in X : q = \sum_{i=1}^{N} q_i e_n, q_i \in \mathbb{Q} + i\mathbb{Q}, N \in \mathbb{N} \right\}.$$

Here $\mathbb{Q} + i\mathbb{Q} := \{z \in \mathbb{C} : z = p + iq, p, q \in \mathbb{Q}\} \subset \mathbb{C}$, which is a dense countable subset of $\mathbb{C}$. Moreover, for each fixed N we can define the subset,

$$Y_N = \left\{ q \in X : q = \sum_{i=1}^{N} q_i e_n, q_i \in \mathbb{Q} + i\mathbb{Q} \right\},$$

and note that there exists a bijection $\varphi : Y_N \to (\mathbb{Q} + i\mathbb{Q})^N$ given by

$$\sum_{i=1}^{N} q_i e_i \mapsto (q_1, \ldots, q_N).$$

Since any finite Cartesian product of countable sets are countable, it follows that $(\mathbb{Q} + i\mathbb{Q})^N$ is countable, and so is Y_N as there is a bijection between them. This implies that the set Y is countable, as

$$Y = \bigcup_{N \in \mathbb{N}} Y_N,$$

and any countable union of countable sets are countable.

It only remains to show that Y is dense in X. Since $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, and thus a finite combination, such that

$$\left\| x - \sum_{i=1}^{N} x_i e_i \right\| < \frac{\varepsilon}{2}.$$

Since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$, there exists a sequence $(q_1, \ldots, q_N) \in (\mathbb{Q} + i\mathbb{Q})^N$ such that for each $1 \leq i \leq N$,

$$|x_i - q_i| < \frac{\varepsilon}{2N \max_{1\leq l\leq N} \|e_l\|}.$$

It therefore follows by the triangle inequality that,

$$\begin{aligned}
\left\| x - \sum_{i=1}^{N} q_i e_i \right\| &\leq \left\| x - \sum_{i=1}^{N} x_i e_i \right\| + \left\| \sum_{i=1}^{N} q_i e_i - \sum_{i=1}^{N} x_i e_i \right\| \\
&< \frac{\varepsilon}{2} + \max_{1\leq j\leq N} \|e_j\| \sum_{i=1}^{N} |x_i - q_i| \\
&< \frac{\varepsilon}{2} + \max_{1\leq j\leq N} \|e_j\| \frac{N\varepsilon}{2N \max_{1\leq l\leq N} \|e_l\|} \\
&= \varepsilon,
\end{aligned}$$

which shows that Y is a countable dense subset in X, and so X is separable.

3 Let $L^2[-1,1]$ be equipped with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\, dt.$$

Apply Gram-Schmidt's orthogonalization algorithm to the monomial basis $\{1, x, x^2, \cdots\}$ up to degree 2.

We begin with the basis elements $\{1, x, x^2\}$, and let $\tilde{e}_j$ denote the j orthogonal element from the Gram-Schmidt procedure, and $e_j = \tilde{e}_j/\|\tilde{e_j}\|$ is the orthonomal element.

For $\tilde{e}_1 = 1$, we simply have

$$e_1 = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{-1} 1^2 dt}} = \frac{1}{\sqrt{2}}.$$

Applying Gram-Schmidt to the vector x gives,

$$\tilde{e}_2 = x - \langle x, e_1\rangle e_1 = x - \frac{1}{2}\int_{-1}^{1} t dt = x.$$

Thus, the nomralized element is given by,

$$e_2 = \frac{x}{\sqrt{\int_{-1}^{1} t^2 dt}} = \frac{x}{\sqrt{\frac{2}{3}}}.$$

For the third and final element, Gram-Schmidt yields

$$\tilde{e}_3 = x^2 - \frac{1}{2}\int_{-1}^{1} t^2 dt - \frac{3}{2}\int_{-1}^{1} t^3 dt x = x^2 - \frac{1}{3}.$$

The normalized element is therefore given as

$$e_3 = \frac{x^2 - 3^{-1}}{\sqrt{\int_{-1}^{1}(t^2 - 3^{-1})^2 dt}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}}.$$

The orthonormal elements up to degree 2 from the Gram-Schmidt procedure are given by

$$\{e_1, e_2, e_3\} = \left\{\frac{1}{\sqrt{2}}, \frac{x}{\sqrt{\frac{2}{3}}}, \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}}\right\}.$$

4 Let $||\cdot||_a$ and $||\cdot||_b$ be equivalent norms on a vector space X. Show that a sequence $\{x_n\}$ in X is Cauchy with respect to the norm $||\cdot||_a$ if and only if it is Cauchy with respect to the norm $||\cdot||_b$.

Recall that two norms are equivalent if and only if there exists $C_1, C_2 > 0$ such that

$$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a,$$

for every $x \in X$.

Assume first that $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_a)$. This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_a < \frac{\varepsilon}{C_2},$$

for every $m, n > N$. However, by the equivalence of the norms, it immediately follows that

$$\|x_n - x_m\|_b \leq C_2\|x_n - x_m\|_a < \varepsilon,$$

and so $\{x_n\}$ is also a Cauchy sequence in $(X, \|\cdot\|_b)$.

Similarly, assume that $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_b)$. This means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_m\|_a < C_1\varepsilon,$$

for every $m, n > N$. However, by the equivalence of the norms, it immediately follows that

$$\|x_n - x_m\|_a \leq \frac{1}{C_1}\|x_n - x_m\|_b < \varepsilon,$$

and so $\{x_n\}$ is also a Cauchy sequence in $(X, \|\cdot\|_a)$.