Norwegian University of Science and Technology
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TMA4145
Linear Methods
Fall 2022

Solutions to exercise set 11

Below follows *one* possible solution to the exercise set.

$\boxed{1}$ Let M be a subspace of an inner product space X. Show that the orthogonal complement $M^\perp$ is closed.

Let us start by showing that $M^\perp$ is a subspace of H. This follows from the linearity of the inner product, and the fact that $0 \perp H$. Namely, for any $x, y \in M^\perp$ and $z \in M$, and $c \in \mathbb{C}$, we have

$$\begin{aligned} \langle x+y, z\rangle &= \langle x, z\rangle + \langle y, z\rangle = 0, \\ \langle cx, z\rangle &= c\langle x, z\rangle = 0, \\ \langle 0, z\rangle &= 0, \end{aligned}$$

which shows that $M^\perp$ is closed under vector addition, scalar multiplication and contains the zero-element. As such, $M^\perp$ is a subspace of H.

To show that $M^\perp$ is a closed, we need to show that if a sequence $\{x_n\}_{n\in\mathbb{N}} \subset M^\perp$ converges to some $x \in H$, then $x \in A^\perp$. Let us therefore consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subset M^\perp$, such that $x_n \to x$ as $n \to \infty$. Note that for each $y \in M$, and each $n \in \mathbb{N}$,

$$|\langle x, y\rangle| = |\langle x, y\rangle - \langle x_n, y\rangle| = |\langle x - x_n, y\rangle| \leq \|x - x_n\|\|y\|,$$

by the Cauchy-Schwarz' inequality and the fact that $x_n \in M^\perp$ for all $n \in \mathbb{N}$. Since $x_n \to x$ in $(H, \langle\cdot,\cdot\rangle)$, we can for every $\varepsilon > 0$, and each $y \in M$ find $N \in \mathbb{N}$ such that

$$\|x - x_n\| < \frac{\varepsilon}{\|y\|},$$

whenever $n > N$. It therefore follows that

$$|\langle x, y\rangle| \leq \|x - x_n\|\|y\| < \varepsilon,$$

by choosing n sufficiently large. This shows that

$$\langle x, y\rangle = 0,$$

and so $x \in M^\perp$, which confirms that $M^\perp$ is closed.

$\boxed{2}$ Let M be the plane of $\mathbb{R}^3$ given by $x_1 + x_2 + x_3 = 0$. Find the linear mapping that is the orthogonal projection of $\mathbb{R}^3$ onto this plane.

In order to find the orthogonal projection, we need an orthonormal basis of the plane as the orthogonal projection is given by projecting onto the orthonormal basis elements. Let us therefore start by finding a basis for the plane M. Rewriting the equation, it follows that $x_3 = -x_1 - x_2$, and so every element in $x \in M$ can be written on the form

$$x = \begin{pmatrix} s \\ t \\ -s-t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

We have therefore a basis for the plane given by

$$E_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Unfortunately, these basis vectors are not orthogonal, as

$$\langle E_1, E_2 \rangle = 1.$$

However, we can utilize the Gram–Schmidt procedure to produce an orthonormal basis. Let us start by defining $e_1 = E_1/\|E_1\|$, namely

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Now, we can define the vector

$$u_2 = E_2 - \langle E_2, e_1 \rangle e_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}.$$

Note that $\langle e_1, u_2 \rangle = 0$, and that e_1 and u_2 is a basis for M. We can therefore define $e_2 = u_2/\|u_2\|$,

$$e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix},$$

and then $\{e_1, e_2\}$ is an orthonormal basis of M.

The projection of an element $x \in \mathbb{R}^3$ onto the plane M is now given by

$$Px = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 = \frac{1}{6}\begin{pmatrix} 4x_1 - 2x_2 - 2x_3 \\ -2x_1 + 4x_2 - 2x_3 \\ -2x_1 - 2x_2 + 4x_3 \end{pmatrix}.$$

For a more explicit form, we note that the projection can be written quite nicely in matrix form by using outer products of vectors. We define the outer product $u \otimes v$ by

$$(u \otimes v)(x) = \langle v, x \rangle u.$$

In the case of $\mathbb{R}^3$, the outer product is simply given by $u \otimes v = uv^T$. This means that the Projection matrix is given by

$$P = e_1e_1^T + e_2e_2^T = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}.$$

$\boxed{3}$ Let $T : X \to X$ be a bounded linear operator on a Hilbert space X. Show that

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

Let us start by showing that $\|T^*T\| = \|T\|^2$. By the definition of the operator norm, it follows that, for each $x \in X$

$$\|T^*Tx\| = \|T^*(Tx)\| \leq \|T^*\|\|Tx\| \leq \|T^*\|\|T\|\|x\| = \|T\|^2\|x\|,$$

where we have used the fact that $\|T\| = \|T^*\|$. Taking the supremum of all $x \in X$, with $\|x\| = 1$, yields

$$\|T^*T\| \leq \|T\|^2.$$

For the reverse inequality, we note that for any $x \in X$

$$0 \leq \|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = |\langle x, T^*Tx\rangle| \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2,$$

where we have used Cauchy-Schwarz' inequality. In particular, this means that

$$\frac{\|Tx\|}{\|x\|} \leq \|T^*T\|^{\frac{1}{2}}.$$

Taking the supremum over all non-zero elements of X yields $\|T\| \leq \sqrt{\|T^*T\|}$, or equivalently

$$\|T\|^2 \leq \|T^*T\|.$$

Thus, we must have $\|T^*T\| = \|T\|^2$.

Finally, we show that $\|T^*T\| = \|TT^*\|$. Let $A = T^*$, and note that by the argument above, and using that $T^{**} = T$, we get

$$\|TT^*\| = \|A^*A\| = \|A\|^2 = \|T^*\|^2 = \|T\|^2 = \|T^*T\|,$$

which concludes the proof.

$\boxed{4}$ Let M be a closed subspace of a Hilbert space X, which by the projection theorem is given by the direct sum $X = M \oplus M^\perp$. Show that the projection onto M is self-adjoint.

By the projection theorem, we can write any element in $x \in H$ as $x = p+e$ where $p \in M$ and $e \in M^\perp$. Moreover, denote the projection onto M by $P_M(x) = P_M(p+e) = p$. Now, for any $x, y \in H$, let $x = p_x + e_x$ and $y = p_y + e_y$ where $p_x, p_y \in M$ and $e_x, e_y \in M^\perp$. Then

$$\begin{aligned}
\langle P_M(x), y\rangle =& \langle p_x, p_y + e_y\rangle \\
=& \langle p_x, p_y\rangle + \langle p_x, e_y\rangle \\
=& \langle p_x, p_y\rangle \\
=& \langle p_x, p_y\rangle + \langle e_x, p_y\rangle \\
=& \langle p_x + e_x, p_y\rangle \\
=& \langle x, P_M(y)\rangle.
\end{aligned}$$

Here we used the fact that $M \perp M^\perp$. This shows that the projection on M is self-adjoint.

$\boxed{5}$ Show that $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$ is orthonormal with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt.$$

For any $n \in \mathbb{Z}$, define the function

$$e_n(t) := e^{2\pi int}.$$

Then for any $n, m \in \mathbb{Z}$, the inner product becomes

$$\langle e_n, e_m\rangle = \int_0^1 e^{2\pi int}e^{-2\pi imt}\,dt = \int_0^1 e^{2\pi i(n-m)t}\,dt.$$

So if $n = m$, then $n - m = 0$ and so,

$$\langle e_n, e_n\rangle = \int_0^1 1\,dt = 1,$$

while for $n \neq m$,

$$\langle e_n, e_n\rangle = \int_0^1 e^{2\pi i(n-m)t}\,dt = \frac{e^{2\pi i(n-m)} - 1}{2\pi i(n-m)} = 0,$$

as $n-m \in \mathbb{Z}$, and the complex exponential is 2π periodic. Combining both equations gives,

$$\langle e_n, e_m\rangle = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

This shows that $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal with given inner product.