Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 9

Below follows *one* possible solution to the exercise set.

1 Given normed spaces X and Y, and a bounded linear operator $A : X \to Y$, show that $\ker(A)$ is a closed subspace of X.

Let us first see that $\ker(A)$ is a subspace of X. This follows from the fact that A is linear. Namely, take $x, y \in \ker(A)$ and $a, b \in \mathbf{F}$. Since $x, y \in \ker(A)$, we have $A(x) = A(y) = 0$, and it follows that

$$A(ax + by) = aA(x) + bA(y) = 0,$$

so $ax + by \in \ker(A)$. Moreover, we have $0 \in \ker(A)$. This follows from

$$A(0) = A(0 + 0) = A(0) + A(0),$$

which can only hold if $A(0) = 0$. This shows that $\ker(A)$ is a subspace.

To show that $\ker(A)$ is closed, we consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \ker(A)$ such that $x_n \to x \in X$, and show that this necessarily implies $x \in \ker(A)$.

Recall that $\|A\| < \infty$, and moreover assume that $\|A\| > 0$ (otherwise, A is just the zero operator). We have $x_n \to x \in X$, so for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$\|x - x_n\|_X < \frac{\varepsilon}{\|A\|},$$

whenever $n > N$. It follows that

$$\|Ax\|_Y = \|Ax - Ax_n\|_Y = \|A(x - x_n)\|_Y \leq \|A\|\|x - x_n\|_X < \varepsilon$$

for every $n > N$. Note that we have used that $x_n \in \ker(A)$, as well as the linearity of A and the definition of the operator norm. Since we can choose $\varepsilon > 0$ arbitrarily small, we conclude that $\|Ax\|_Y = 0$ and thus $Ax = 0$. This shows that $x \in \ker(A)$, so $\ker(A)$ is closed.

2 Let T be the integral operator

$$Tf(x) = \int_0^1 k(x, y)f(y)dy,$$

defined by a kernel $k \in C([0, 1] \times [0, 1])$ such that $k(x, y) \geq 0$ for any $(x, y) \in [0, 1] \times [0, 1]$. Show that the operator norm of T as a mapping on $C[0, 1]$ with respect to the $\|\cdot\|_\infty$-norm is

$$\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x, y)|dy.$$

Let us start by showing that the right-hand side is an upper bound for the operator norm. Recall that the operator norm is defined as

$$\|T\| = \sup_{\|f\|_\infty = 1} \|Tf\|_\infty.$$

Thus, let $f \in C([0,1])$ such that $\|f\|_\infty = 1$. Then, by the triangle inequality for integrals,

$$\left| \int_0^1 k(x,y)f(y)dy \right| \leq \int_0^1 |k(x,y)f(y)|dy \leq \|f\|_\infty \int_0^1 |k(x,y)|dy = \int_0^1 |k(x,y)|dy.$$

In particular this means that

$$\|Tf\|_\infty = \max_{x\in[0,1]} \left| \int_0^1 k(x,y)f(y)dy \right| \leq \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy,$$

and so we have an upper bound for $\|Tf\|_\infty$, when $\|f\|_\infty = 1$. Since the supremum is the least upper bound, it follows that

$$\|T\| = \sup_{\|f\|_\infty = 1} \|Tf\|_\infty \leq \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy.$$

To show the other inequality, consider the function $f(x) = 1$ for all $x \in [0,1]$. This function satisfies $\|f\|_\infty = 1$, and

$$Tf(x) = \int_0^1 k(x,y)dy = \int_0^1 |k(x,y)|dy,$$

as $k(x,y) \geq 0$ for all $(x,y) \in [0,1] \times [0,1]$. In particular, we have

$$\max_{x\in[0,1]} \int_0^1 |k(x,y)|dy = \|Tf\|_\infty \leq \|T\|.$$

This proves that

$$\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy.$$

3 Let M be a closed subspace of a Hilbert space H, and let P be the orthogonal projection of H onto M. Prove that P is bounded and linear, and find $\|P\|$. Is P isometric?

As M is a closed subspace, we can decompose $H = M \bigoplus M^\perp$. This means that every $x \in H$ can be written as $x = p + e$, where $p \in M$ and $e \in M^\perp$. The orthogonal projection is the unique operator $P : H \to M$, which maps $x \mapsto p = P(x)$. Moreover, $P = P^2 = P^*$.

To show linearity, we let $x, y \in H$, and $a, b \in \mathbb{F}$. By the decomposition of H, we have $x = p_x + e_x$, and $y = p_y + e_y$. Since M is a closed subspace, it follows that $ap_x + bp_y \in M$, while $ae_x + be_y \in M^\perp$. Thus it follows that

$$P(ax + by) = P(a(p_x + e_x) + b(p_y e_y)) = ap_x + bp_y = aP(x) + bP(y),$$

which shows that P is linear.

To prove that P is bounded, we will use that $P = P^2 = P^*$. Thus we have

$$\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle P^2x, x\rangle = \langle Px, x\rangle \le \|Px\|\|x\|,$$

where we used Cauchy-Schwarz' inequality at the end. This shows that

$$\|Px\| \le \|x\|.$$

In particular,

$$\|P\| = \sup_{x\in H:\|x\|=1} \|Px\| \le \|x\| = 1.$$

This shows that P is bounded, and the operator norm is bounded by 1. On the other hand, note that for any $x \in M$, we have $Px = x$, and so

$$\|Px\| = \|x\|,$$

which shows that $\|P\| = 1$.

The orthogonal projection on a closed *proper* subspace of H is not isometric. Take any $x \in M^\perp \setminus \{0\}$. We then have $Px = 0$ (since $x \in M^\perp$), and

$$\|x\| \ne 0 = \|Px\|,$$

so P is not isometric.

4 Define an operator $B : \ell^1 \to \ell^1$ by

$$Bx = \left(\frac{x_k}{k}\right)_{k\in\mathbb{N}} = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right), \quad x = (x_k)_{k\in\mathbb{N}} \in \ell^1.$$

a) Show that B linear, bounded, and $\|B\| = 1$.

Let us start by showing that B is linear. Consider two sequence $x, y \in \ell^1$, and $a, b \in \mathbb{F}$. Then

$$\begin{aligned} B(ax+by) &= \left(ax_1 + by_1, \frac{ax_2+by_2}{2}, \frac{ax_3+by_3}{3}, \dots\right) \\ &= a\left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right) + b\left(y_1, \frac{y_2}{2}, \frac{y_3}{3}, \dots\right) \\ &= aBx + bBy, \end{aligned}$$

which shows that B is linear. Moreover, note that for any $x \in \ell^1$, we have

$$\|Bx\|_1 = \sum_{k=1}^{\infty} \frac{|x_k|}{k} \le \sum_{k=1}^{\infty} |x_k| = \|x\|_1,$$

so B is bounded, with operator norm $\|B\| \le 1$.

Now, consider $\delta_1 = (1, 0, 0, \dots) \in \ell^1$. Then $B\delta_1 = (1, 0, 0, \dots) = \delta_1$, and

$$\|B\delta_1\|_1 = \|\delta_1\|_1 = 1.$$

This shows that $\|B\| = 1$.

b) Show that B is injective, but not surjective.

To show that B is injective, we need to show that if there exists $x, y \in \ell^1$, such that $Bx = By$, then $x = y$.

Let us therefore assume that there are two sequences $x, y \in \ell^1$ such that $Bx = By$. Then

$$(0,0,0,\ldots) = Bx - By = \left(x_1 - y_1, \frac{x_2 - y_2}{2}, \frac{x_3 - y_3}{3}, \ldots \right) = \left(\frac{x_k - y_k}{k} \right)_{k \in \mathbb{N}},$$

which shows that $x_k = y_k$ for all $k \in \mathbb{N}$. We conclude that $x = y$, which shows that B is injective.

An operator $T : X \to Y$ is surjective if for each $y \in Y$, there exists $x \in X$ such that $Tx = y$. For our case, this means that if B is surjective, then for each $y \in \ell^1$, there exists $x \in \ell^1$ such that $Bx = y$. In particular

$$\left(\frac{x_k}{k} \right)_{k \in \mathbb{N}} = Bx = y = (y_k)_{k \in \mathbb{N}},$$

which implies that $x_k = k y_k$, for all $k \in \mathbb{N}$. Now, consider the series $y = (y_k)_{k \in \mathbb{N}} = (k^{-2})_{k \in \mathbb{N}} \in \ell^1$. Then $x = (k y_k)_{k \in \mathbb{N}} = (k^{-1})_{k \in \mathbb{N}} \notin \ell^1$. This shows that B is not surjective, as there is no element $x \in \ell^1$ which maps to $y = (y_k)_{k \in \mathbb{N}} = (k^{-2})_{k \in \mathbb{N}} \in \ell^1$.

c) Prove that range(B) is a proper dense subspace of ℓ^1, but it is not closed.

It follows from the linearity of B that range(B) is a subspace of ℓ^1. That it is a proper subspace follows from **b)**, as $y = (y_k)_{k \in \mathbb{N}} = (k^{-2})_{k \in \mathbb{N}} \notin \text{range}(B)$.

It remains to show that range(B) is a dense subspace, and that it is not closed. Note, however, that if we can prove that range(B) is dense we also know that range(B) is not closed; if range(B) is dense, then $\overline{\text{range}(B)} = \ell^1$. Thus, for every $x \in \ell^1$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \text{range}(B)$ such that $x_n \to x$. However, we know that range(B) is proper, so there exists $x \in \ell^1$ such that $x \notin \text{range}(B)$, and thus range(B) cannot be closed.

To prove that range(B) is dense, we note that $c_{00} \subseteq \text{range}(B)$, where c_{00} is the set of sequences with only finitely many non-zero terms,

$$c_{00} := \{(x_k)_{k \in \mathbb{N}} : \exists\, N \in \mathbb{N} \text{ such that } x_n = 0, \forall n > N\}.$$

To see this, simply observe that for any $x = (x_k)_{k \in \mathbb{N}} \in c_{00}$, we can construct the sequence $z = (k x_k)_{k \in \mathbb{N}}$, and $Bz = x$. The sequence z is clearly an element of ℓ^1, since

$$\|z\|_1 = \sum_{k=1}^{\infty} k|x_k| = \sum_{k=1}^{N} k|x_k| \leq N^2 \max_{1 \leq k \leq N} |x_k| < \infty.$$

Now let us show that c_{00} is dense in ℓ^1. Take any $x = (x_k)_{k \in \mathbb{N}} \in \ell^1$, and consider the truncated sequence $\{x^n\}_{n \in \mathbb{N}} = \{(x_k^n)_{k \in \mathbb{N}}\}_{n \in \mathbb{N}} \subset c_{00}$, given by

$$x^n = (x_1, \ldots, x_n, 0, 0, \ldots).$$

We claim that $x^n \to x$. To see this, note that

$$\|x - x^n\|_1 = \sum_{k=1}^{\infty} |x_k - x_k^n| = \sum_{k=n+1}^{\infty} |x_k|,$$

and since $x \in \ell^1$, we know that

$$\sum_{k=n+1}^{\infty} |x_k| \to 0 \quad \text{as} \quad n \to \infty.$$

This verifies that $x^n \to x$ in ℓ^1. Since $x \in \ell^1$ was arbitrary, it follows that c_{00} is dense in ℓ^1. However, we also have that $c_{00} \subseteq \text{range}(B)$, and so it follows that

$$\overline{c_{00}} = \overline{\text{range}(B)} = \ell^1.$$

This shows that range(B) is a proper dense subspace of ℓ^1 which is not closed.

5 Let X, Y, Z be vector spaces and $T : X \to Y$ and $S : Y \to Z$ be linear transformations. Which of the following statements are true?

1. This is false. Counterexample: Let $T : \mathbb{R} \to \mathbb{R}^2$ be defined as $T(x) = (x, x)$ and $S : \mathbb{R}^2 \to \mathbb{R}$ as $S(x, y) = x + y$. Then $S \circ T(x) = 2x$, which is onto, but T is not onto.
2. $S \circ T$ is surjective implies that S is surjective: This is true. If $S \circ T$ is surjective, then for each $z \in Z$, there exists $x \in X$ such that

$$S \circ T(x) = z.$$

However, as $T : X \to Y$, we note that $T(x) = y \in Y$. This means that for each $z \in Z$ we can find $y \in Y$ such that $S(y) = z$ (specifically, $y = T(x)$). Thus, S is surjective.

3. $S \circ T$ is injective implies that T is injective: This is true. Consider $x_1, x_2 \in X$ such that $T(x_1) = T(x_2) \in Y$. Then

$$S \circ T(x_1) = S \circ T(x_2),$$

and since $S \circ T$ is injective, this implies $x_1 = x_2$. Thus, T is injective.

4. This is false, by the same counterexample as in 1.

Another counterexample that disproves 1. and 4. is the following: consider the transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$, and $S : \mathbb{R}^3 \to \mathbb{R}^2$, given by

$$T(x_1, x_2) = (x_1, x_2, 0) \quad S(x_1, x_2, x_3) = (x_1, x_2).$$

It is clear that T is injective, but not surjective, while S is surjective, but not injective. On the other hand $S \circ T = I_2$ is the identity on $\mathbb{R}^2$, which is bijective.

6 Let X be a Banach space, and let $A \in \mathcal{B}(X)$ be given. Let $A^0 = I$ be the identity map on X, and denote by A^n the composition $A^n = A \circ \ldots \circ A$ n-times.

a) Prove that the series

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

converges absolutely in operator norm, and $\|e^A\| \leq e^{\|A\|}$.

The series converges absolutely if

$$\lim_{N\to\infty} \sum_{k=0}^{N} \frac{\|A^k\|}{k!} < \infty.$$

We first show that $\|A^k\| \leq \|A\|^k$ for every $k \in \mathbb{N}$ using induction. This is clearly true for $k = 2$, as for any $x \in X$, we have

$$\|A^2 x\|_X \leq \|A\|\|Ax\|_X \leq \|A\|^2\|x\|_X \quad \Longrightarrow \quad \|A^2\| \leq \|A\|^2,$$

by the definition of the operator norm. Assume now that it holds for the case $k-1$. We observe that

$$\|A^k x\|_X = \|A(A^{k-1}x)\|_X \leq \|A\|\|A^{k-1}x\|_X \leq \|A\|^k\|x\|_X,$$

and thus $\|A^k\| \leq \|A\|^k$. If we use this fact, we see that

$$\lim_{N\to\infty} \sum_{k=0}^{N} \frac{\|A^k\|}{k!} \leq \lim_{N\to\infty} \sum_{k=0}^{N} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty,$$

and so the series converges absolutely with respect to the operator norm.

Additionally: Note that the argument above implies that the partial sums are Cauchy in $\mathcal{B}(X)$, as for $n > m$ we have

$$\left\| \sum_{k=0}^{n} \frac{A^k}{k!} - \sum_{k=0}^{m} \frac{A^k}{k!} \right\| = \left\| \sum_{k=m+1}^{n} \frac{A^k}{k!} \right\| \leq \sum_{k=m+1}^{n} \frac{\|A\|^k}{k!}.$$

Since the last sum converges in $\mathbb{R}$, it follows by Cauchy's convergence criterion that for each $\varepsilon > 0$, the exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} \frac{\|A\|^k}{k!} < \varepsilon,$$

whenever $n > m > N$. This implies that the partial sums are Cauchy in $\mathcal{B}(X)$, so the partial sums converge to a unique element in $\mathcal{B}(X)$ (since $\mathcal{B}(X)$ is complete).

To show that $\|e^A\| \leq e^{\|A\|}$, note that by continuity of the norm, we get

$$\|e^A\| = \left\| \lim_{N\to\infty} \sum_{k=0}^{N} \frac{A^k}{k!} \right\| = \lim_{N\to\infty} \left\| \sum_{k=0}^{N} \frac{A^k}{k!} \right\| \leq \lim_{N\to\infty} \sum_{k=0}^{N} \frac{\|A\|^k}{k!} = e^{\|A\|}.$$

b) Prove that for each $x \in X$, we have

$$e^A x = \sum_{k=0}^{\infty} \frac{A^k x}{k!},$$

where the series converges absolutely with respect to the norm on X.

We use a similar argument as above. Namely, we have

$$\sum_{k=0}^{N} \frac{\|A^k x\|_X}{k!} \leq \sum_{k=0}^{N} \frac{\|A\|^k}{k!} \|x\|_X,$$

and taking the limit on both sides as $N \to \infty$, we get

$$\sum_{k=0}^{\infty} \frac{\|A^k x\|_X}{k!} \leq e^{\|A\|} \|x\|_X < \infty.$$

This shows that the series converges absolutely. By the same argument as above, the partial sums give rise to a Cauchy sequence in X, and so the series converges to a unique element in X.