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Solutions to exercise set 3

Below follows *one* possible solution to the exercise set.

1 **a)** Consider the metric space $(\mathbb{Z}, d)$ where $d(x, y) = |x - y|$ is the usual metric. We want to give an example of a Cauchy sequence, and to show that this metric space is complete.

Fix $q \in \mathbb{Z}$, and define the constant sequence $x_n = q$ for all n. This is necessarily a Cauchy sequence, as for any $m, n \in \mathbb{Z}$, and all $\varepsilon > 0$ we have

$$|x_n - x_m| = |q - q| = 0 < \varepsilon.$$

This is an example of a Cauchy sequence in $(\mathbb{Z}, d)$.

In order to show that this space is complete, we have to show that any Cauchy sequence converges to an element in $\mathbb{Z}$ with respect to the metric d. Let us first make a crucial observation. Namely, for any $q, p \in \mathbb{Z}$ where $q \neq p$, we have

$$|q - p| \geq 1.$$

Let $\{q_n\}_{n=0}^{\infty}$ be a Cauchy sequence in $(\mathbb{Z}, d)$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|q_n - q_m| < \varepsilon,$$

for all $n, m > N$. In particular, by choosing $\varepsilon = 1/2$, we can find N such that for all $m, n > N$

$$|q_n - q_m| < \frac{1}{2}.$$

However, by the observation made above this can only happen if $q_n = q_m$ for all $n, m > N$. Let us denote this constant integer $q = q_n \in \mathbb{Z}$ for all $n > N$. It then follows that for any $n > N$

$$|q - q_n| = |q - q| = 0,$$

and so the Cauchy sequence converges to an integer. We therefore conclude that the space is complete, as the choice of Cauchy sequence was arbitrary.

What we have observed above is that any Cauchy sequence in $(\mathbb{Z}, d)$ necessarily has to have a constant tail.

b) Consider the metric space $(\mathbb{R}, d)$, where

$$d(x, y) = |\arctan(x) - \arctan(y)|.$$

Is this a complete metric space.

We claim that $(\mathbb{R}, d)$ is not a complete metric space. To see why, we note that the function $\arctan(x)$ is a strictly increasing function which is bounded from above by $\pi/2$. Moreover, since

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2},$$

we can, for every $\varepsilon > 0$, find an $x_\varepsilon \in \mathbb{R}$ such that

$$\left|\frac{\pi}{2} - \arctan(x)\right| = \frac{\pi}{2} - \arctan(x) < \varepsilon,$$

for all $x \geq x_\varepsilon$.

Now choose a strictly increasing sequence $\{x_i\}_{i\in\mathbb{N}}$, such that

$$\lim_{i\to\infty} x_i = \infty \notin \mathbb{R}.$$

Then for every $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that $x_n > x_\varepsilon$ for all $n > N$. Since $\arctan(x)$ is a strictly increasing function, it follows that $\arctan(x_\varepsilon) < \arctan(x_n) < \pi/2$ for all $n > N$. Therefore, if we choose $n, m > N$, we have

$$d(x_n, x_m) = |\arctan(x_n) - \arctan(x_m)| < \frac{\pi}{2} - \arctan(x_\varepsilon) < \varepsilon,$$

and thus the sequence $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy. On the other hand, we know that our sequence does not converge to an element in $\mathbb{R}$. This shows that $(\mathbb{R}, d)$ is not a complete metric space.

Alternatively: We can also show this more directly. We claim that the sequence $\{n\}_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{R}, d)$. Namely, for $m > n$

$$\begin{aligned}
\arctan(m) - \arctan(n) &= \int_n^m \frac{1}{x^2+1} dx \\
&\leq \int_n^m \frac{1}{x^2} dx \\
&= \frac{1}{n} - \frac{1}{m} \leq \frac{1}{n}.
\end{aligned}$$

In particular, for any $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon,$$

and so for any $m > n > N$

$$d(n, m) \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

On the other hand, the sequence $n \to \infty$, and so is an example of a Cauchy sequence which does not converge in $\mathbb{R}$.

$\boxed{2}$ Let ℓ^∞ be the vector space of bounded real-valued sequences

$$\ell^\infty = \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\| = \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}.$$

a) Show that d_∞, defined by $d_\infty(x,y) = \|x-y\|_\infty$ is a metric on ℓ^∞.

Recall that there are four criteria which need to hold for d_∞ to be a metric. These are non-negativity, uniqueness, symmetry and the triangle inequality. Let us start by showing non-negativity. Given $x, y \in \ell^\infty$, we have

$$d_\infty(x,y) = \|x-y\|_\infty = \sup_{n\in\mathbb{N}} |x_n - y_n| \geq |x_i - y_i| \geq 0,$$

for all $i \in \mathbb{N}$, by the definition of the supremum.

For uniqueness, we note that given $x, y \in \ell^\infty$ such that $d_\infty(x,y) = 0$, we must have

$$0 = \sup_{n\in\mathbb{N}} |x_n - y_n| \geq |x_i - y_i| \geq 0,$$

for all $i \in \mathbb{N}$. However, this implies that $x_i = y_i$ for all $i \in \mathbb{N}$, and so $x = y$. The symmetry follows easily from the symmetry of the absolute value, namely

$$d_\infty(x,y) = \sup_{n\in\mathbb{N}} |x_n - y_n| = \sup_{n\in\mathbb{N}} |y_n - x_n| = d_\infty(y,x).$$

To show the triangle inequality, we choose $x, y, z \in \ell^\infty$. For any $i \in \mathbb{N}$, we then have

$$|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|.$$

By taking the supremum on both sides it follows that

$$\begin{aligned} d_\infty(x,y) \leq & \sup_{n\in\mathbb{N}} \left(|x_n - z_n| + |z_n - y_n|\right) \\ \leq & \sup_{n\in\mathbb{N}} |x_n - z_n| + \sup_{n\in\mathbb{N}} |z_n - y_n| \\ = & d_\infty(x,z) + d_\infty(z,y). \end{aligned}$$

This shows that d_∞ is a metric on ℓ^∞.

b) Show that ℓ^∞ is complete with respect to d_∞.

To show that (ℓ^∞, d_∞) is a complete metric space, we have to show that every Cauchy sequence converges to an element in ℓ^∞.

In order to easier keep track of indices during the proof, we will denote sequences in ℓ^∞ by a superscript. That is, we denote a sequence of sequences in ℓ^∞ by $\{x^i\}_{i\in\mathbb{N}}$, where each element in the sequence is given by

$$x^i = (x_1^i, x_2^i, \ldots) \in \ell^\infty.$$

Now let $\{x^i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in (ℓ^∞, d_∞). Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_\infty(x^i, x^j) = \sup_{n\in\mathbb{N}} |x_n^i - x_n^j| < \varepsilon,$$

whenever $i, j > N$. This implies that for every $m \in \mathbb{N}$,

$$|x_m^i - x_m^j| < \varepsilon,$$

and $(x_m^i)_{i\in\mathbb{N}}$ is therefore a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Since the latter metric space is complete, there exists $x_m \in \mathbb{R}$ such that $\lim_{i\to\infty} x_m^i = x_m$, for all $m \in \mathbb{N}$.

This gives us a candidate for the limit of $\{x^i\} \in \ell^\infty$, namely

$$x = (x_1, x_2, \ldots).$$

Let us start by showing that x is a limit point of x^i. For every $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $i > N$ we have

$$|x^i_m - x_m| < \varepsilon,$$

for all $m \in \mathbb{N}$. This follows from the fact that $x^i_m \to x_m$. Moreover, this implies that

$$d_\infty(x, x^i) = \sup_{n \in \mathbb{N}} |x^i_n - x_n| < \varepsilon,$$

as the supremum is the *least* upper bound. This shows that $x^i \to x$ with respect to d_∞.
To show that $x \in \ell^\infty$, we use that $\{x^i\}_{i \in \mathbb{N}}$ is Cauchy. Namely we can find an N such that $d_\infty(x, x^N) < 1$. Whence it follows that

$$\|x\|_\infty = d_\infty(x, 0) \leq d_\infty(x, x^N) + d_\infty(x^N, 0) < 1 + \|x^N\|_\infty < \infty,$$

as $x^N \in \ell^\infty$. This shows that $x \in \ell^\infty$. Moreover, since the Cauchy sequence $\{x^i\}_{i \in \mathbb{N}} \subset \ell^\infty$ was arbitrary, it follows that every Cauchy sequence in (ℓ^∞, d_∞) converges to an element in ℓ^∞, and thus (ℓ^∞, d_∞) is complete.

$\boxed{3}$ Show that every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (X, d) is bounded.

A set E is bounded in a metric space if there exists $x \in X$ and $r > 0$ such that

$$E \subseteq B_r(x) = \{y \in X : d(x, y) < r\}.$$

This is definition 2.3.2 on page 53 in Heil's book.

Since the sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy, we know that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon,$$

for all $n, m > N$.

Let us choose $\varepsilon = 1$. Then there is a constant N_1 such that

$$d(x_n, x_m) < 1,$$

for all $n, m > N_1$. Now define

$$M := \max_{1 \leq n \leq N_1} d(x_n, x_{N_1+1}).$$

We claim that

$$\{x_n\}_{n \in \mathbb{N}} \subseteq B_{M+1}(x_{N_1+1}).$$

If we can show this inclusion, we have proven that the sequence is bounded.

Note that every element $x_n \in B_{M+1}(x_{N_1+1})$ for $1 \leq n \leq N_1$. To see why, we have for all $1 \leq i \leq N_1$,

$$d(x_i, x_{N_1+1}) < \max_{1 \leq n \leq N_1} d(x_n, x_{N_1+1}) = M < M + 1.$$

Moreover, $x_{N_1+1} \in B_{M+1}(x_{N_1+1})$ as $d(x_{N_1+1}, x_{N_1+1}) = 0 < M + 1$.

For the rest of the sequence, we will use that the sequence is Cauchy. Namely for any $n > N_1 + 1$, we have

$$d(x_n, x_{N_1+1}) < 1 \leq 1 + M.$$

This proves that

$$\{x_n\}_{n\in\mathbb{N}} \subseteq B_{M+1}(x_{N_1+1}),$$

and so we have shown that every Cauchy sequence in a metric space is bounded.

$\boxed{4}$ Let $X = \mathbb{R}$, and find the interior E°, the boundary, ∂E and the closure $\overline{E}$ of each of the subsets.

a) We consider the following set

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}.$$

Note that for any $r > 0$, and $n \in \mathbb{N}$, we can find

$$x \in B_r\left(\frac{1}{n}\right) = \left\{y \in \mathbb{R} : \left|\frac{1}{n} - y\right| < r\right\},$$

such that $x \notin E$. In particular, there are no open balls $B_r(n^{-1})$ which are contained in E, and so the only open set contained in E is the empty set $\emptyset$. Since E° is the largest open set contained in E, it follows that $E^\circ = \emptyset$.

The boundary ∂E is the set of all boundary points of E. A boundary point $x \in E$ is such that for any $r > 0$

$$B_r(x) \cap E \neq \emptyset \quad B_r(x) \cap E^C \neq \emptyset.$$

This is definition 2.5.4 on page 64 in Heil's book.

By the same argument as above we see that for any $r > 0$ and $n \in \mathbb{N}$ we have $E \cap B_r(n^{-1}) \neq \emptyset$ and $E^C \cap B_r(n^{-1}) \neq \emptyset$. This means that

$$E \subseteq \partial E.$$

Note that for any $r > 0$, there exists $n \in \mathbb{N}$ such that $n^{-1} < r$, and so

$$\frac{1}{n} \in B_r(0).$$

This means that 0 is also a boundary point.

There cannot be any other boundary points. If this was the case, then there would be an $x \in \mathbb{R}$ such that for any $r > 0$ we can find $n_r^{-1} \in E$ satisfying

$$\left|x - \frac{1}{n_r}\right| < r.$$

This means that n_r^{-1} converges to x as $r \to 0$. However, the only limit points of sequences in E are either in E or 0 as the sequences have either constant tails or are subsequences of n^{-1} which converge to 0. We can therefore conclude that

$$\partial E = E \cup \{0\}.$$

The closure of a set is the collection of all limit points of E by Theorem 2.6.2 on page 67 in Heil's book. By the previous argument $\overline{E} = E \cup \{0\}$.
To summarize
$$E^\circ = \emptyset, \quad \partial E = \overline{E} = E \cup \{0\}.$$

b) Consider the set $E = [0, 1)$. The largest open set contained in E is the open interval $(0, 1)$, so $E^\circ = (0, 1)$.
The boundary points of E are the points 0 and 1, as these are the only two points p for which
$$B_r(p) \cap [0, 1) \neq \emptyset \quad \text{and} \quad B_r(p) \cap [1, \infty) \neq \emptyset,$$
for any $r > 0$. We write $\partial E = \{0, 1\}$.
For the closure, we note that $[0, 1]$ is the smallest closed set which contains E. It therefore follows from definition 2.6.1 on page 66 of Heil's book that
$$\overline{E} = [0, 1].$$
To summarize
$$E^\circ = (0, 1), \quad \partial E = \{0, 1\} \quad \overline{E} = [0, 1].$$

c) Consider $E = \mathbb{R}\backslash\mathbb{Q}$, the set of irrational numbers. Note that each irrational $x \in E$ can be approximated arbitrarily well by a rational number. This implies that for no $x \in E$ can we find $r > 0$ such that $B_r(x) \subset E$, which in turn implies that
$$E^\circ = \emptyset.$$
Likewise, for every point $x \in \mathbb{R}$ and any $r > 0$ there exists $q \in \mathbb{Q}$ and $y \in E$ such that $q, y \in B_r(x)$. We thus have
$$\partial E = \mathbb{R}.$$
For the closure of E we use the fact that any real number can be written as the limit of a sequence of irrational numbers. Hence, we have
$$\overline{E} = \mathbb{R}.$$
To summarize
$$E^\circ = \emptyset, \quad \partial E = \overline{E} = \mathbb{R}.$$

$\boxed{5}$ Let us consider the following subsets of ℓ^∞.
$$\begin{aligned} c_{00} =& \{x = (x_1, \ldots, x_N, 0 \ldots) : N > 0, x_1, \ldots, x_N \in \mathbb{R}\} \\ c_0 =& \{x \in \ell^\infty : \lim_{k\to\infty} x_k = 0\}. \end{aligned}$$

a) By theorem 2.6.2, we have that the closure of a set is the collection of all its limit points. That is
$$\overline{c_{00}} = \{y \in \ell^\infty : \exists \{x^n\}_{n\in\mathbb{N}} \subset c_{00} \text{ such that } \lim_{x\to\infty} x^n = y\}.$$

Let us first show that $\overline{c_{00}} \subset c_0$. Let $\{x^n\}_{n\in\mathbb{N}}$ be a sequence converging to $y \in \ell^\infty$. Then for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$\|x^n - y\|_\infty = \sup_{m\in\mathbb{N}} |x^n_m - y_m| < \varepsilon,$$

when $n > N$. However, since $x^n \in c_{00}$, it follows that there is some $M > 0$ such that

$$|x^n_m - y_m| = |y_m| < \varepsilon,$$

for $m > M$. This implies that

$$\lim_{m\to\infty} |y_m| = 0,$$

and $\overline{c_{00}} \subseteq c_0$.

Now take any $y = (y_1, y_2, \ldots) \in c_0$, then we can define the sequence

$$x^n = (y_1, y_2, \ldots, y_n, 0, 0 \ldots).$$

Since $\lim_{m\to\infty} y_m = 0$, it follows that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|y_m| < \varepsilon,$$

for all $m > N$.

In particular, we have for $n > N$,

$$\|y - x^n\|_\infty = \sup_{m>n} |y_m| < \varepsilon,$$

and so y is the limit point of x^n, and hence $y \in \overline{c_{00}}$. This shows that $c_0 \subseteq \overline{c_{00}}$. We can therefore conclude that $c_0 = \overline{c_{00}}$.

b) We want to show that c_0 is closed in (ℓ^∞, d).

This follows directly from **a)**, as $c_0 = \overline{c_{00}}$, and the closure of a set E is the smallest *closed* set containing E. This is definition 2.6.1 on page 66 in Heil's book.

Alternatively: This can also be shown directly. By theorem 2.4.2 on page 59, a set E is closed if and only if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $x_n \to x \in X$, we have $x \in E$.

Let $\{x^n\}_{n\in\mathbb{N}} \subset c_0$ be a sequence converging to $y \in \ell^\infty$. Then for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$, and $M \in \mathbb{N}$ such that

$$\begin{cases} \|x^n - y\|_\infty = \sup_{i\in\mathbb{N}} |x^n_i - y_i| < \frac{\varepsilon}{2} & n > N, \\ |x^n_m| < \frac{\varepsilon}{2}, & m > M. \end{cases}$$

The first inequality follows from the fact that $x^n \to y$, while the second follows as $x^n \in c_0$. However, this means that for $n > N$ and $m > M$, we have

$$|y_m| \le |y_m - x^n_m| + |x^n_m| < \varepsilon,$$

and so $y_m \to 0$ as $m \to \infty$, and $y \in c_0$. It follows from theorem 2.4.2 that c_0 is closed.

6 **Challenge:** Let E be a subset of a metric space (X, d). Prove that E is dense in X if and only if $E \cap U \neq \emptyset$ for every non-empty open set $U \subset X$.

Recall that a set E is called dense if $\overline{E} = X$. By theorem 2.6.2 the closure of the set E is the set of all limit points of E.

($\Rightarrow$)Let us start by assuming that E is dense. Then for every element $x \in X$ there exists sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $x_n \to x$ in (X, d).

Let $U \subset X$ be a non-empty open set. Since U is non-empty, there exists $x \in X$ such that $x \in U$. Moreover, U is open. This means that there exists $r > 0$ such that $B_r(x) \subset U$. Since E is dense, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ and an $N_r \in \mathbb{N}$ such that

$$d(x, x_n) < r,$$

for all $n > N_r$ However, this implies that $x_n \in B_r(x) \subset U$, and since $x_n \in E$, we must have $E \cap U \neq \emptyset$. Since U was arbitrary, we conclude that $E \cap U \neq \emptyset$ for all open sets $U \subset X$.

($\Leftarrow$) Assume that $E \cap U \neq \emptyset$ for every open set $U \subset X$. In particular, this is true for any open ball in X. Thus for every $x \in X$, and every $n \in \mathbb{N}$, we have

$$B_{\frac{1}{n}}(x) \cap E \neq \emptyset.$$

Now fix any $x \in X$, and construct a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that for any $n \in \mathbb{N}$, we have

$$x_n \in B_{\frac{1}{n}}(x) \cap E.$$

This sequence converges to x. To see this, observe that for every $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $N^{-1} < \varepsilon$. In particular we have,

$$d(x, x_n) < \frac{1}{n} < \frac{1}{N} < \varepsilon,$$

for $n > N$ as $x_n \in B_{n^{-1}}(x)$. This shows that any x is a limit point of E, and thus E is dense in X.