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TMA4145
Linear Methods
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Solutions to exercise set 11

Below follows *one* possible solution to the exercise set.

1 We will prove the statement by induction. If $k = 1$ we only have one vector v_1, and since we assume that v_1 is an eigenvector, we know that v_1 is non-zero. Hence the set $\{v_1\}$ is linearly independent; if $c_1 v_1 = 0$ for some scalar c_1 and non-zero vector v_1, then clearly $c_1 = 0$.

Now assume that the statement is true for $k-1$ eigenvalues, and assume that

$$c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0 \tag{1}$$

where c_j are scalars and v_j are non-zero eigenvectors of an operator T corresponding to distinct eigenvalues λ_j. We need to show that $0 = c_1 = c_2 \cdots = c_k$. Since $Tv_k = \lambda_k v_k$, we have that $(T - \lambda_k I)v_k = 0$, where I is the identity operator. We can therefore get rid of v_k in equation (1) by applying $T - \lambda_k I$ to the equation. We then get

$$0 = (T - \lambda_k I)(c_1 v_1 + \cdots c_k v_k) = c_1(\lambda_1 - \lambda_k)v_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1},$$

where we have used that the v_j are eigenvectors. By the induction assumption, $\{v_1, v_2, ..., v_{k-1}\}$ is a linearly independent set. Therefore

$$0 = c_1(\lambda_1 - \lambda_k) = c_2(\lambda_2 - \lambda_k) = ... = c_{k-1}(\lambda_{k-1} - \lambda_k),$$

and since we assume that the eigenvalues are distinct this implies that

$$0 = c_1 = c_2 = ... = c_{k-1}.$$

Therefore equation (1) reads

$$c_k v_k = 0,$$

and as before this implies that also $c_k = 0$, hence $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set.

2 Let T be the linear operator on the space of polynomials $\mathcal{P}_2$ of degree at most 2 defined by $Tf(x) = -f(x) - f'(x)$

a) In the basis $\{1, x, x^2\}$, we can write any polynomial $f(x) = a + bx + cx^2$, where $a, b, c \in \mathbb{R}$. Thus, when applying the operator T, we see that

$$Tf(x) = -f(x) - f'(x) = -(cx^2 + bx + a) - (2cx + b) = -(a+b) - (b+2c)x - cx^2.$$

The matrix representation of T in the basis $\{1, x, x^2\}$ is then given by

$$T = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}.$$

b) Assume that there exists some $\lambda \in \mathbb{R}$ such that $Tf = \lambda f$. Then we have

$$Tf = -f - f' = \lambda f \quad \Longrightarrow \quad f' = -(1+\lambda) f,$$

which has the solution $f(x) = Ce^{-(1+\lambda)x} \notin \mathcal{P}_2$, except for $\lambda = -1$, where the solution is a constant. Thus, the operator T has only one eigenvalue, $\lambda = -1$, and corresponding eigenvectors are the constant functions.

3

1. Assume that λ is an eigenvalue of T with eigenvector $y = (y_1, y_2, \dots)$. Then $Ty = \lambda y$, and writing out both sides we find

$$(0, y_1, y_2, \dots) = (\lambda y_1, \lambda y_2, \lambda y_3 \dots). \tag{2}$$

In particular $\lambda y_1 = 0$, which implies that either $y_1 = 0$ or $\lambda = 0$. If $\lambda = 0$, then equality (2) becomes

$$(0, y_1, y_2, \dots) = (0, 0, \dots)$$

which shows that $y = 0$, hence not an eigenvector since eigenvectors are non-zero by definition. We may therefore assume that $y_1 = 0$ and $\lambda \neq 0$. In this case the equality (2) becomes

$$(0, 0, y_2, \dots) = (0, \lambda y_2, \lambda y_3 \dots), \tag{3}$$

which implies that $y_2 = 0$. Inserting this back into the equation, we find

$$(0, 0, 0, \dots) = (0, 0, \lambda y_3 \dots), \tag{4}$$

hence $y_3 = 0$. We may clearly continue like this to show that all entries of y are 0, hence $y = 0$ and y is not an eigenvector.

2. We know from the lectures (and it is not difficult to show) that $T^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$. This operator has eigenvalues. For instance, let

$$y = \left(1, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{p-1}}, \dots\right).$$

Then

$$T^* y = \left(\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^p}, \dots\right) = \frac{1}{2} y,$$

hence y is an eigenvector with eigenvalue $\frac{1}{2}$. Note that $y \in \ell^2$, which we needed since we defined T on ℓ^2.

4 Note that the column u_i is of the form $u_i = (u_{1,i}, u_{2,i}, \ldots, u_{n,i})^T$, and the inner product between two columns is given by

$$\langle u_i, u_j \rangle = \sum_{k=1}^{n} u_{k,i}\overline{u_{k,j}} = u_i \cdot u_j^*,$$

where $u_i \cdot u_j^*$ is the usual dot product for vectors in $\mathbb{C}^n$ and

$$u_j^* = \begin{pmatrix} \overline{u_{1,j}} \\ \overline{u_{2,j}} \\ \cdots \\ \overline{u_{n,j}} \end{pmatrix}.$$

$\underline{1 \Rightarrow 2}$

Assume that $U^*U = I$. Recall that element (i, j) of the matrix product U^*U is the dot product of row i of U^* with column j of U. Since row i of U^* is u_i^*, this means that element (i, j) of U^*U is $u_i^* \cdot u_j$. Furthermore $U^*U = I$, which implies that $u_i^* u_j = \delta_{i,j}$ for $i, j = 1, \ldots, n$. We thus have

$$\langle u_j, u_i \rangle = u_i^* \cdot u_j = \delta_{i,j},$$

hence $(u_1, u_2, \ldots, u_n)$ is an orthonormal system of vectors in $\mathbb{C}^n$. To show that it is a basis for $\mathbb{C}^n$ it is enough to note that $\mathbb{C}^n$ has dimension n, and the system consists of n vectors. Hence the columns form a linearly independent subset of n vectors in an n-dimensional space, and it follows that the columns form a basis.

$\underline{2 \Rightarrow 1}$

Assume that the columns $u_1, u_2, \ldots, u_n$ of U are an orthonormal basis of $\mathbb{C}^n$, i.e.

$$\langle u_i, u_j \rangle = \delta_{i,j},$$

for $i, j = 1, \ldots, n$. Then we have

$$u_i \cdot u_j^* = \langle u_i, u_j \rangle = \delta_{i,j},$$

hence we have $U^*U = I$. One gets that $U^*U = I$ from exactly the same argument.

5 If $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is normal, the it can be diagonalized by the spectral theorem. Namely, there exists a unitary matrix U, and a diagonal matrix Λ, with the eigenvalues of A on the diagonal, such that

$$A = U\Lambda U^*.$$

In particular, since $\det(AB) = \det(A)\det(B)$, we have

$$\begin{aligned}
\det(A) &= \det(U\Lambda U^*) \\
&= \det(U)\det(\Lambda)\det(U^*) \\
&= \det(U)\det(U^*)\det(\Lambda) \\
&= \det(UU^*)\det(\Lambda) \\
&= \det(I)\det(\Lambda) \\
&= \det(\Lambda) = \prod_{j=1}^{n} \lambda_j,
\end{aligned}$$

where we used that the determinant of a diagonal matrix is the product of the diagonal elements, and that $\det(I) = 1$.

6 Find the general solution to the system of differential equations

$$x_1' = x_1 + x_3,$$
$$x_2' = x_2 + x_3,$$
$$x_3' = 2x_3.$$

The system can be written as

$$x' = Ax,$$

where the vector x', and matrix A is given by,

$$x' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

A general solution $x \in \mathbb{R}^n$ to the equation

$$x' = Ax,$$

is given by

$$x = \sum_{j=1}^{n} c_j v_j e^{\lambda_j t}, \quad c_j \in \mathbb{C}.$$

where λ_j are the eigenvalues of A, and v_j are the corresponding eigenvectors.

In our case, the matrix A is upper triangular, and so the eigenvalues are given by $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$, which are the diagonal entries of A.

Let us therefore find the eigenvectors corresponding to these eigenvalues. For $\lambda_1 = \lambda_2 = 1$, we have the eigenvectors,

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_3 = 2$, we have the eigenvector,

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Combining these results, we see that the general solution to the differential equation is given by

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}.$$

7 Consider the matrix

$$A = \begin{pmatrix} -1.8 & 0 & -1.4 \\ -5.6 & 1 & -2.8 \\ 2.8 & 0 & 2.4 \end{pmatrix}.$$

We want to diagonalize it, and find A^m for $m \in \mathbb{N}$. Let us therefore start by finding the eigenvalues of A, by solving

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 1)(\lambda + 0.4) = 0,$$

in order to see that the eigenvalues are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = -0.4$. For the eigenvectors, we find that

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

Let V and Σ be the matrices

$$V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5} \end{pmatrix}.$$

then the inverse matrix is given by

$$V^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix},$$

and so the diagonalization of A is given by

$$A = V\Sigma V^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix}.$$

To calculate A^m, for $m \in \mathbb{N}$, we note that

$$A^2 = V\Sigma V^{-1} V\Sigma V^{-1} = V\Sigma^2 V^{-1}.$$

Thus, a simple induction argument shows that

$$A^m = V\Sigma^m V^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(-\frac{2}{5}\right)^m \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix}.$$

Moreover, the entry-wise limit $\lim_{m\to\infty} A^m$ does exists, and is given by

$$\lim_{m\to\infty} A^m = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ -2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -2 \\ 2 & 0 & 2 \end{pmatrix}.$$