Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 10

Below follows *one* possible solution to the exercise set.

1 Define the linear transformation T by

$$T\left(\sum_{k=1}^{n} c_k v_k\right) = (c_1 + c_2)v_1 + \ldots + (c_{n-1} + c_n)v_{n-1} + c_n v_n.$$

T is a linear transformation. More generally, for any $u_j \in V$, there is a unique linear transformation T such that $T(v_j) = u_j$.

The columns of the matrix representation of T in the basis $\{v_1, \ldots, v_n\}$ are the coefficients of $T(v_1), \ldots, T(v_n)$ in this basis. Let A denote this matrix representation. Then A is on the form

$$A = \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

2 We will consider $\mathbb{R}^3$ with the usual inner product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Moreover, we let $\|\cdot\|$ be the norm induced by the inner product, and d denote the metric induced by the norm.

Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix.

$$A = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}.$$

Determine if the following statements are true.

1. This is true; T **is** a self-adjoint operator. Since A is a symmetric, real matrix, we have that
$$A^* = A^\top = A,$$
so the corresponding operator is self-adjoint.

2. This is true; T **is** a normal operator. Recall that T is normal if $TT^* = T^*T$. This corresponds to showing that $A^*A = AA^*$. We have already seen that $A^* = A$ in 1., and it follows that

$$A^*A = AA = AA^*,$$

so A is a normal matrix, and thus T is a normal operator.

3. This is false; T **is not** a unitary operator. For an operator T to be unitary, we need that $TT^* = T^*T = I$. However, a simple calculation shows that

$$A^*A = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 & \frac{1}{8} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

This shows that A is not unitary, hence neither is T.

4. This is true; T **is** a contraction on the metric space $(\mathbb{R}^3, d)$. For $x \in \mathbb{R}^3$, we have

$$Tx = Ax = \begin{pmatrix} \frac{1}{4}x_1 + \frac{1}{4}x_3 \\ \frac{1}{2}x_2 \\ \frac{1}{4}x_1 + \frac{1}{4}x_3 \end{pmatrix} = x_1 \begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}.$$

Thus, we can see that for $x, y \in \mathbb{R}^3$,

$$\begin{aligned} \|Tx - Ty\|^2 = & \left(\frac{1}{4}(x_1 - y_1) + \frac{1}{4}(x_3 - y_3)\right)^2 + \frac{1}{4}(x_2 - y_2)^2 \\ & + \left(\frac{1}{4}(x_1 - y_1) + \frac{1}{4}(x_3 - y_3)\right)^2 \\ = & \frac{1}{8}\left((x_1 - y_1) + (x_3 - y_3)\right)^2 + \frac{1}{4}(x_2 - y_2)^2 \\ \leq & \frac{3}{8}(x_1 - y_1)^2 + \frac{1}{4}(x_2 - y_2)^2 + \frac{3}{8}(x_3 - y_3)^2 \\ \leq & \frac{3}{8}\|x - y\|^2. \end{aligned}$$

Here we used the hint that $(a + b)^2 \leq 3(a^2 + b^2)$ for all $a, b \in \mathbb{R}$. Taking the square root, we see that

$$\|Tx - Ty\| \leq \sqrt{\frac{3}{8}}\|x - y\|, \quad \forall x, y \in \mathbb{R}^3. \tag{1}$$

This shows that T is a contraction in $(\mathbb{R}^3, d)$, with contraction constant

$$c = \sqrt{\frac{3}{8}} < 1.$$

5. This is false; The operator norm of T is $\|T\| = \sup_{\|x\|=1} \|Ax\| \neq 1$. This follows from the previous result. Inserting $y = 0$ in (1), we get

$$\|Tx\| = \|Tx - 0\| \leq \sqrt{\frac{3}{8}}\|x - 0\| = \sqrt{\frac{3}{8}}\|x\|,$$

and taking the supremum over all x with $\|x\| = 1$, it follows that

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \leq \sqrt{\frac{3}{8}} < 1.$$

3 Let $T : X \to X$ be a bounded linear operator on a Hilbert space X. Show that

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

Let us start by showing that $\|T^*T\| = \|T\|^2$. By the definition of the operator norm, it follows that, for each $x \in X$

$$\|T^*Tx\| = \|T^*(Tx)\| \leq \|T^*\|\|Tx\| \leq \|T^*\|\|T\|\|x\| = \|T\|^2\|x\|,$$

where we have used the fact that $\|T\| = \|T^*\|$. Taking the supremum of all $x \in X$, with $\|x\| = 1$, yields

$$\|T^*T\| \leq \|T\|^2.$$

For the reverse inequality, we note that for any $x \in X$

$$0 \leq \|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = |\langle x, T^*Tx\rangle| \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2,$$

where we have used Cauchy-Schwarz' inequality. In particular, this means that

$$\frac{\|Tx\|}{\|x\|} \leq \|T^*T\|^{\frac{1}{2}}.$$

Taking the supremum over all non-zero elements of X yields $\|T\| \leq \sqrt{\|T^*T\|}$, or equivalently

$$\|T\|^2 \leq \|T^*T\|.$$

Thus, we must have $\|T^*T\| = \|T\|^2$.

Finally, we show that $\|T^*T\| = \|TT^*\|$. Let $A = T^*$, and note that by the argument above, and using that $T^{**} = T$, we get

$$\|TT^*\| = \|A^*A\| = \|A\|^2 = \|T^*\|^2 = \|T\|^2 = \|T^*T\|,$$

which concludes the proof.

4 Let X_1, and X_2 be two Hilbert spaces and $T \in \mathcal{B}(X_1, X_2)$.

a) Show that there exists $T^* \in \mathcal{B}(X_2, X_1)$ such that

$$\langle Tx, y\rangle_{X_2} = \langle x, T^*y\rangle_{X_1}.$$

For each fixed $y \in X_2$, we can define the linear functional, $l_y : X_1 \to \mathbb{C}$, by

$$l_y(x) := \langle Tx, y\rangle_{X_2}.$$

By Cauchy-Schwarz' inequality and the fact that $T \in \mathcal{B}(X_1, X_2)$, we have

$$|l_y(x)| = |\langle Tx, y\rangle_{X_2}| \leq \|Tx\|_{X_2}\|y\|_{X_2} \leq \|T\|\|y\|_{X_2}\|x\|_{X_1},$$

and since T and y are fixed, it follows that l_y is a bounded linear functional. Thus, by Riesz' representation theorem, there exists a unique $z_y \in X_1$ such that

$$\langle Tx, y\rangle_{X_2} = l_y(x) = \langle x, z_y\rangle_{X_1}.$$

Note that this holds for any $y \in X_2$, so we define the map $T^* : X_2 \to X_1$ by $y \mapsto z_y$. By definition, we then have

$$\langle x, T^* y\rangle_{X_1} = \langle x, z_y\rangle_{X_1} = l_y(x) = \langle Tx, y\rangle_{X_2}.$$

We need to show that T^* is linear and bounded. To show linearity, let $y, \eta \in X_2$, and $a, b \in \mathbb{C}$. Then for each $x \in X_1$, we have

$$\begin{aligned}\langle x, T^*(ay + b\eta)\rangle_{X_1} =&\langle Tx, ay + b\eta\rangle_{X_2}\\ =&\overline{a}\langle Tx, y\rangle_{X_2} + \overline{b}\langle Tx, \eta\rangle_{X_2}\\ =&\overline{a}\langle x, T^* y\rangle_{X_1} + \overline{b}\langle x, T^*\eta\rangle_{X_1}\\ =&\langle x, aT^* y + bT^*\eta\rangle_{X_1}.\end{aligned}$$

Since this holds for all $x \in X_1$, we conclude that $T^*(ay + b\eta) = aT^*y + bT^*\eta$, which shows that T^* is linear.

It only remains to show that T^* is bounded. Recall (from problem 1, exercise 7) that for a Hilbert space X, and any $x \in X$

$$\|x\|_X = \sup_{\|y\|=1} |\langle x, y\rangle_X|$$

Whence it follows that for any $y \in X_2$, we have

$$\|T^* y\|_{X_1} = \sup_{\|x\|=1} |\langle x, T^* y\rangle_{X_1}| = \sup_{\|x\|=1} |\langle Tx, y\rangle_{X_2}| \leq \sup_{\|x\|=1} \|Tx\|_{X_2}\|y\|_{X_2} = \|T\|\|y\|_{X_2},$$

where we first used Cauchy-Schwarz' inequality, and then the definition of the operator norm of T. This confirms that T^* is bounded, and thus $T^* \in \mathcal{B}(X_2, X_1)$.

b) Show that $\ker T = \ker T^*T$.

One inclusion follows quite easily. Namely, if $x \in \ker T$, then $Tx = 0$, and so

$$T^*Tx = T^*(0) = 0,$$

as T^* is a linear transformation. This shows that $\ker T \subseteq \ker T^*T$.

For the other inclusion, we let $x \in \ker T^*T$. We then have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = 0.$$

By the uniqueness of the norm, it follows that $Tx = 0$, and hence $x \in \ker T$. Since x was any element in $\ker T^*T$, this shows that $\ker T \subseteq \ker T^*T$. Hence, we have shown that $\ker T = \ker T^*T$.

5 **a)** Let M be a closed subspace of a Hilbert space H. For each $x \in H$, denote by $P_M(x)$ the orthogonal projection of x onto M. Prove that $P_M^2 = P_M$, and $P_M^* = P_M$.

As M is a closed subspace of H, we can decompose $H = M \bigoplus M^\perp$, in the sense that we have a unique representation of each $x \in H$ given by

$$x = p_x + e_x, \quad p_x \in M, e_y \in M^\perp,$$

and $P_M(x) = p_x$. Note that this necessarily means that for any $x \in H$,

$$P_M^2(x) = P_M(p_x) = p_x = P_M(x).$$

So, $P_M^2 = P_M$. Moreover, for any $x, y \in H$, we have

$$\langle x, P_M^*(y)\rangle = \langle P_M(x), y\rangle = \langle p_x, p_y + e_y\rangle = \langle p_x, p_y\rangle = \langle p_x + e_x, p_y\rangle = \langle x, P_M(y)\rangle,$$

where we have used the linearity of the inner product, and the fact that $e_x, e_y \in M^\perp$. This shows that $P_M^* = P_M$, and thus concludes the proof.

b) Consider the bounded operator $T_a : \ell^2 \to \ell^2$ given by

$$T_a(x) = (a_1x_1, a_2x_2, \ldots),$$

for a fixed $a = (a_i)_{i\in\mathbb{N}} \in \ell^\infty$. Show that the condition $a_i \in \{0, 1\}$ is necessary for T_a to be an orthogonal projection on a closed subspace of ℓ^2. Verify that this is also sufficient by showing that $\ker(T_a)^\perp = \mathrm{range}(T_a)$ under this condition.

We have seen in **a)** that for T_a to be an orthogonal projection, we must have $T_a^* = T_a$ and $T_a^2 = T_a$. The latter condition means that for any $x \in \ell^2$, we must have

$$0 = T_a^2x - T_ax = (a_1^2x_1 - a_1x_1, a_2^2x_2 - a_2x_2, \ldots) = (a_i(a_1 - 1)x_i)_{i\in\mathbb{N}}\,.$$

Since this should hold for any $x \in \ell^2$, we must have

$$a_i(a_i - 1) = 0, \quad \Longrightarrow \quad a_i = 0 \text{ or } a_i = 1,$$

for every $i \in \mathbb{N}$. Under this condition, we also have that $T_a^* = T_{\overline{a}} = T_a$.

Let us now verify that $\ker(T_a)^\perp = \mathrm{range}(T_a)$ when $a_i \in \{0, 1\}$ for all i. Let $y \in \ker(T_a)$. Then for any $x \in \ell^2$,

$$\langle T_ax, y\rangle = \sum_{i=1}^{\infty} a_ix_i\overline{y_i} = \sum_{i=1}^{\infty} x_i\overline{a_iy_i} = \langle x, T_ay\rangle = 0,$$

This shows that $T_ax \in \ker(T_a)^\perp$, so $\mathrm{range}(T_a) \subseteq \ker(T_a)^\perp$. For the reversed inclusion, let $J = \{j \in \mathbb{N} : a_j = 0\}$, and note that

$$\ker T_a = \{y \in \ell^2 : y_j = 0 \text{ for all } j \notin J\},$$

and

$$\text{range } T_a = \{y \in \ell^2 : y_j = 0 \text{ for all } j \in J\}.$$

In particular, any standard unit vector δ_{j_0} with $j_0 \in J$ lies in $\ker T_a$. Now let $x \in \ker(T_a)^\perp$. We then have

$$\langle x, \delta_{j_0}\rangle = x_{j_0} = 0, \qquad \text{for all } j_0 \in J.$$

This shows that $x \in \text{range } T_a$, so $\ker(T_a)^\perp \subseteq \text{range } T_a$, and we conclude that $\ker(T_a)^\perp = \text{range } T_a$.

An alternative approach is to use that for a bounded operator T on a Hilbert space H, we have

$$\ker(T^*)^\perp = \mathrm{range}(T),$$

and since $T_a = T_a^*$, we get

$$\ker(T_a)^\perp = \ker(T_a^*)^\perp = \mathrm{range}(T_a).$$

c) Determine the operator norm $\|T_a\|$ for a general fixed $a \in \ell^\infty$.
We claim that $\|T_a\| = \|a\|_\infty$. Note that for any $x \in \ell^2$, we have

$$\|T_a x\|_2^2 = \sum_{i=1}^{\infty} |a_i x_i|^2 \leq \|a\|_\infty^2 \sum_{i=1}^{\infty} |x_i|^2 = \|a\|_\infty^2 \|x\|_2^2.$$

Taking the square root, and then the supremum of all $x \in \ell^2$ with $\|x\| = 1$, we have

$$\|T_a\| \leq \|a\|_\infty.$$

On the other hand, if we denote the standard basis of ℓ^2 by $\{\delta_k\}_{k\in\mathbb{N}}$, we see that

$$\|T_a \delta_k\|_2 = |a_k|, \quad \forall k \in \mathbb{N}.$$

Since $\|\delta_k\|_2 = 1$ for all $k \in \mathbb{N}$, it follows that

$$|a_k| \leq \|T_a\| \quad \text{for any } k \in \mathbb{N}.$$

Taking the supremum over all k, and using the fact that the supremum is the least upper bound, it follows that

$$\|a\|_\infty = \sup_{k\in\mathbb{N}} |a_k| \leq \|T_a\|,$$

and hence $\|a\|_\infty = \|T_a\|$.

6 **a)** Let $T : B(V) \to B(V)$ be defined by $T(X) = I + AX$. Then

$$\|T(X) - T(Y)\| = \|A(X - Y)\| \leq \|A\|\|X - Y\|.$$

Since $\|A\| < 1$, this shows that T is a contraction. Since V is a Banach space, it follows that $B(V)$ is a Banach space. This means that $B(V)$ is complete, and thus by Banach's fixed point theorem there exists a unique operator $X \in B(V)$ such that $X = T(X) = I + AX$, which means that

$$(I - A)X = I.$$

b) The fixed point can be found by iteration. Let $X_0 = 0$. Then $X_1 = T(X_0) = I$ and $X_2 = T(X_1) = I + A$. Suppose that $X_n = I + A + A^2 + \ldots + A^{n-1}$. Then

$$X_{n+1} = T(X_n) = I + AX_n = I + A + A^2 + \ldots + A^n = \sum_{k=0}^{n} A^k.$$

By Banach's fixed point theorem, we know that the sequence $\{X_n\}$ converges to the fixed point X, so $X = \sum_{k=0}^{\infty} A^k$.

We know that $(I - A)X = I$, but we also need to show that $X(I - A) = I$. Note that $AX_n = X_n A$ for each $n \in \mathbb{N}$. Thus,

$$AX = AX_n + A(X - X_n) = XA + (X_n - X)A + A(X - X_n),$$

and letting $X_n \xrightarrow{n\to\infty} X$ we see that $AX = XA$. This implies that

$$X(I - A) = X - XA = X - AX = (I - A)X,$$

and so X is the inverse of $I - A$.