Exercise set 6

Norwegian University of Science and Technology
Department of Mathematical Sciences

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 The space of continuous functions that *vanish at infinity* is

$$C_0(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) \, : \, \lim_{t \to \pm\infty} f(t) = 0 \right\}.$$

Prove that $C_0(\mathbb{R})$ is a closed subspace of $C_b(\mathbb{R})$ with respect to the uniform norm $\|\cdot\|_u$, and therefore $(C_0(\mathbb{R}), \|\cdot\|_u)$ is a Banach space.

2 Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent norms on a vector space X, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that (x_n) is a Cauchy sequence in $(X, \|\cdot\|_a)$ if and only if (x_n) is a Cauchy sequence in $(X, \|\cdot\|_b)$.

3 Show that the norm $\|\cdot\|_p$ on ℓ^p does not satisfy the parallelogram law

$$\|x - y\|_p^2 + \|x + y\|_p^2 = 2\left(\|x\|_p^2 + \|y\|_p^2\right) \qquad \text{for all } x, y \in X,$$

for any $p \neq 2$. Thus, $\|\cdot\|_p$ is not induced by an inner product for any $p \neq 2$.

4 Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space.

a) Show that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in X$, then $x = y$.

b) Given $x_n, y_n, x, y \in X$, show that if $x_n \to x$ and $y_n \to y$, then

$$\langle x_n, y_n \rangle \to \langle x, y \rangle.$$

5 Suppose $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, and let $\|x\| = \langle x, x \rangle^{1/2}$.

a) Show that $\|\cdot\|$ satisfies the parallelogram law.

b) Let ω be a n^{th} root of unity, meaning $\omega^n = 1$ and $\omega^k \neq 1$ for $k < n$. Show that for $n \geq 3$, we have

$$\langle x, y \rangle = \frac{1}{n} \sum_{k=1}^{n} \omega^k \|x + \omega^k y\|^2.$$

c) Show that

$$\langle x, y \rangle = \int_0^1 e^{2\pi i\varphi} \|x + e^{2\pi i\varphi} y\|^2 \, d\varphi.$$

6 A $d \times d$ matrix A with real entries is *positive definite* if it is symmetric ($A = A^T$) and $Ax \cdot x > 0$ for all nonzero vectors $x \in \mathbb{R}^d$, where $x \cdot y$ denotes dot product for vectors in $\mathbb{R}^d$.

a) Show that if A is a positive definite $d \times d$ matrix, then

$$\langle x, y \rangle_A = Ax \cdot y, \quad x, y \in \mathbb{R}^d,$$

defines an inner product on $\mathbb{R}^d$.

b) Show that if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^d$, then necessarily there exists some positive definite $d \times d$ matrix A such that

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_A.$$