Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2021

Exercise set 13

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 *(Continuation exam, August 2021)*

a) Show that

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

defines a metric on $\mathbb{R}$. Describe the unit ball $B_1(0)$ in $(\mathbb{R}, d)$.
Hint: Show that the function $f(t) = t/(1+t)$ is increasing for all $t \geq 0$.

b) Is $(\mathbb{R}, d)$ a complete metric space? Either prove this, or provide a counterexample disproving it. You may use (without proof) that $\mathbb{R}$ is complete when equipped with the usual Euclidean distance metric.

2 *(Continuation exam, August 2021)* Consider the real vector space of continuous functions $C[-1,1]$. Show that

$$\|f\| = \int_{-1}^{1} |tf(t)|\, dt, \quad f \in C[-1,1],$$

defines a norm on $C[-1,1]$. Show that this norm is *not* equivalent to the L^1-norm

$$\|f\|_1 = \int_{-1}^{1} |f(t)|\, dt.$$

3 *(Continuation exam, August 2021)* Find $a, b \in \mathbb{C}$ such that

$$\int_0^2 |f(t) - a - bt|^2\, dt$$

is minimal when f is given by

$$f(t) = \begin{cases} \sin \pi t, & 0 \leq t \leq 1. \\ 0, & 1 < t \leq 2 \end{cases}.$$

4 *(Exam 2020, Problem 6)* Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}.$$

a) Determine if A is normal. Find the singular values of A, and determine $\|A\|$ (the norm of the operator $\mathbb{C}^2 \to \mathbb{C}^2$ represented by A).

b) Find a singular value decomposition of A.

c) Find the polar decomposition of A.

5 Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

a) Compute the singular value decomposition of A.

b) Use the result of **a)** to find the pseudoinverse of A, and the best approximation to a solution of $Ax = b$ having minimal norm when $b = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$.