Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2021

Exercise set 10

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let $\{v_1, \dots, v_n\}$ be a basis for a finite-dimensional vector space V. Show that there exists a linear transformation $T : V \to V$ such that $T(v_1) = v_1$, $T(v_j) = v_{j-1} + v_j$, $j = 2, \dots, n$. Find the matrix of this transformation in the basis $\{v_1, \dots, v_d\}$.

2 (*Continuation exam, August 2021, Problem 2 modified*) Consider $\mathbb{R}^n$ with the standard inner product

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n,$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by matrix multiplication $Tx = Ax$, where

$$A = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 0 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \end{bmatrix}.$$

Let $\|\cdot\|$ denote the norm induced by the standard inner product, and let d denote the metric induced by this norm. Determine whether the following statements are true or false (and explain why).

1. T is a self-adjoint operator.
2. T is a normal operator.
3. T is a unitary operator.
4. T is a contraction on the metric space $(\mathbb{R}^3, d)$.
   Hint: Since $ab \le a^2 + b^2$ *for any* $a, b \in \mathbb{R}$, *we have that* $(a+b)^2 \le 3(a^2 + b^2)$.
5. The operator norm of T is $\|T\| = \sup_{\|x\|=1} \|Ax\| = 1$.

3 Let $T : X \to X$ be a bounded linear operator on a Hilbert space X. Show that

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

4 Let X_1 and X_2 be two Hilbert spaces and $T \in B(X_1, X_2)$.

a) Show that there exists $T^* \in B(X_2, X_1)$ such that $\langle Tx, y\rangle_{X_2} = \langle x, T^*y\rangle_{X_1}$ for any $x \in X_1$, $y \in X_2$.
(Note: We treated the case $X_1 = X_2$ in class.)

b) Prove that $\ker T = \ker T^*T$.

5 (*Continuation exam 2018, problem 5*)

a) Let M be a closed subspace of a Hilbert space H. For each $x \in H$ denote by $P_M(x)$ the orthogonal projection of x onto M. Prove that $P_M^2 = P_M$ and $P_M^* = P_M$.

b) Now consider the bounded linear operator $T_a : \ell^2 \to \ell^2$ given by

$$T_a(x) = (a_1x_1, a_2x_2, a_3x_3, \ldots),$$

where $a = (a_1, a_2, \ldots)$ is a fixed element of ℓ^∞. Show that the condition $a_i \in \{0, 1\}$ is necessary for T_a to be an orthogonal projection on a closed subspace of ℓ^2. Verify that this is also sufficient by showing that $\ker(T_a)^\perp = \text{range}(T_a)$ under this condition.

c) Determine the operator norm $\|T_a\|$ (no longer assuming the condition on a given in **b)**).

6 **Challenge:** (*Exam 2002, Problem 2*) Let V be a Banach space, and let $A \in \mathcal{B}(V)$ (i.e. a bounded linear operator on V) with operator norm $\|A\| < 1$. In this problem, we will show that $I - A \in \mathcal{B}(V)$ is invertible by using Banach's fixed point theorem. Here I is the identity operator ($Iv = v$ for all $v \in V$).

Let $T : \mathcal{B}(V) \to \mathcal{B}(V)$ be given by $T(X) = I + AX$ for $X \in \mathcal{B}(V)$.

a) Explain how one can use Banach's fixed point theorem to show that there is one and only one $X \in \mathcal{B}(V)$ such that

$$(I - A)X = I.$$

b) Show by induction that

$$T^{n+1}(0) = I + A + A^2 + \cdots + A^n$$

for $n \geq 0$, and conclude that

$$X = \sum_{k=0}^{\infty} A^k$$

is the fixed point of T. Why do we have that X is the inverse of $I - A$, i.e. that $X(I - A) = (I - A)X = I$?