Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2021

Exercise set 1

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Give an example of an infinite sequence of sets $A_0, A_1, A_2, \ldots$ such that $A_0 = \mathbb{Z}$ and

$$A_0 \supsetneq A_1 \supsetneq A_2 \ldots$$

Remark: This one, as many problems in the course, admits many different solutions. Try to find your own.

2 Let X, Y and Z be sets. Show that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

3 Let $f : X \to Y$ be a function, let A be a subset of X, and let B be a subset of Y.

a) Prove that $A \subseteq f^{-1}(f(A))$, and if f is injective then equality holds. Show by example that equality need not hold if f is not injective.

b) Prove that $f(f^{-1}(B)) \subseteq B$, and if f is surjective then equality holds. Show by example that equality need not hold if f is not surjective.

4 Show that the set $\mathbb{Q}$ of rational numbers is countable.

5 Let $C(\mathbb{R}, \mathbb{R})$ denote the set of real-valued continuous functions on $\mathbb{R}$. Moreover, let $C^1(\mathbb{R}, \mathbb{R}) \subset C(\mathbb{R}, \mathbb{R})$ be the subset of functions with continuous first derivatives. Define a function from the set $\{f \in C^1(\mathbb{R}, \mathbb{R}) : f(0) = 0\}$ to the set $C(\mathbb{R}, \mathbb{R})$ by

$$f \longmapsto f'.$$

Show that this function is a bijection.

6 Find a domain A such that the following functions are bijections between A and $f(A)$. Try to choose A as large as possible. What is $f(A)$ and f^{-1}?

a) $A \subset \mathbb{R}$, $x \longmapsto x^2 - x$

b) $A \subset \mathbb{R}$, $x \longmapsto \cos x$

c) $A \subset \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

d) $A \subset \mathbb{C}$, $z \longmapsto e^z$

7 **(Challenge)** Prove that the closed interval $[0, 1]$ and the open interval $(0, 1)$ have the same cardinality by finding a bijection $f : [0, 1] \to (0, 1)$. *Hint: Do not try to create a continuous function f.*