ADJOINT OPERATORS

Consider a Hilbert space $X$ over a field $F \in \{ \mathbb{R}, \mathbb{C} \}$. In this note we introduce *adjoint operators*, which provide us with an alternative description of bounded linear operators on $X$. We will see that the existence of so-called adjoints is guaranteed by Riesz’ representation theorem.

**Theorem 1** (Adjoint operator). Let $T \in \mathcal{B}(X)$ be a bounded linear operator on a Hilbert space $X$. There exists a unique operator $T^* \in \mathcal{B}(X)$ such that
\[
\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \text{for all } x, y \in X.
\]

The operator $T^*$ is called the *adjoint* of $T$.

**Proof.** Existence: Fix $y \in X$, and define the map
\[
\varphi(x) = \langle Tx, y \rangle, \quad x \in X.
\]
This is a bounded linear functional on $X$, as it is easily seen to be linear and
\[
|\varphi(x)| = |\langle Tx, y \rangle| \leq \|Tx\|\|y\| \leq \|T\|\|y\||x|.
\]
By Riesz’ representation theorem it follows that there exists a unique element $y^* \in X$ such that
\[
\varphi(x) = \langle Tx, y \rangle = \langle x, y^* \rangle \quad \text{for all } x \in X.
\]
We thus define $T^*y := y^*$, so by definition $T^*$ satisfies $\langle Tx, y \rangle = \langle x, T^*y \rangle$. It remains to show that $T^*$ is linear, bounded and unique.

**Linearity of $T^*$:** We have
\[
\langle x, T^* (\alpha y_1 + \beta y_2) \rangle = \langle Tx, \alpha y_1 + \beta y_2 \rangle \\
= \alpha \langle Tx, y_1 \rangle + \beta \langle Tx, y_2 \rangle \\
= \alpha \langle x, T^*y_1 \rangle + \beta \langle x, T^*y_2 \rangle \\
= \langle x, \alpha T^*y_1 + \beta T^*y_2 \rangle \quad \text{for all } x \in X,
\]
and it follows that
\[
T^* (\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2.
\]

**Boundedness of $T^*$:** By the Cauchy-Schwarz inequality, we get
\[
\|T^*y\|^2 = \langle T^*y, T^*y \rangle = \langle TT^*y, y \rangle \\
\leq \|TT^*y\|\|y\| \\
\leq \|T\|\|T^*y\|\|y\|.
\]
If $\|T^*y\| > 0$, we divide by $\|T^*y\|$ on both sides in the inequality and obtain
\[
\|T^*y\| \leq \|T\|\|y\|.
\]
This inequality is clearly also satisfied when \( \|T^*y\| = 0 \), so \( T^* \) is a bounded operator. Moreover, we have attained the additional information
\[
\|T^*\| \leq \|T\|.
\]

**Uniqueness:** Suppose there exists another operator \( S \in \mathcal{B}(X) \) such that
\[
\langle x, Sy \rangle = \langle x, T^*y \rangle \quad \text{for all } x, y \in X.
\]
Then necessarily, for each \( y \in X \), we have
\[
\langle x, Sy - T^*y \rangle = 0 \quad \text{for all } x \in X.
\]
It follows that \( Sy = T^*y \) for every \( y \in X \), meaning \( S = T^* \). \( \square \)

We list and prove some useful properties of adjoints.

**Proposition 2.** Let \( X \) be a Hilbert space, \( S : X \to X \) and \( T : X \to X \) be bounded linear operators and \( \alpha, \beta \in \mathbb{F} \) any two scalars. We then have:

i) \( (\alpha S + \beta T)^* = \alpha S^* + \beta T^* \)

ii) \( (ST)^* = T^* S^* \)

iii) \( (T^*)^* = T \)

iv) \( \|T^*\| = \|T\| \)

v) \( \|TT^*\| = \|T^*T\| = \|T\|^2 \)

**Proof.** i) and ii): Exercise.

iii) Fix any \( y \in X \). We have
\[
\langle x, T^{**}y \rangle = \langle T^*x, y \rangle = \langle y, T^*x \rangle = \langle \overline{y}, x \rangle = \langle x, Ty \rangle
\]
for all \( x \in X \). It thus follows that \( T^{**} = T \).

iv) In the proof of the existence of the adjoint, we established that \( \|T^*\| \leq \|T\| \).
For the opposite inequality, simply observe that by iii), we have
\[
\|T\| = \|T^{**}\| \leq \|T^*\|
\]
and thus \( \|T^*\| = \|T\| \).

v): Exercise on Problem set 11. \( \square \)

**Example 3.**

i) **Left and right shift operators:** Consider the right shift operator \( R \) on \( \ell^2 \), given by
\[
Rx = (0, x_1, x_2, x_3, \ldots), \quad x = (x_j)_{j \in \mathbb{N}} \in \ell^2.
\]

Its adjoint is the left shift operator \( L \), given by
\[
Lx = (x_2, x_3, x_4, \ldots).
\]
To see this, observe that
\[ \langle Rx, y \rangle = \langle (0, x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots) \rangle = x_1 y_2 + x_2 y_3 + x_3 y_4 + \ldots = \langle (x_1, x_2, x_3, \ldots), (y_2, y_3, y_4, \ldots) \rangle = \langle x, Ly \rangle \]
for any \( x, y \in \ell^2 \). Thus, the operator \( R^* \) satisfying \( \langle Rx, y \rangle = \langle x, R^* y \rangle \) for all \( x, y \in \ell^2 \) is \( R^* = L \).

ii) Multiplication operator on \( \ell^2 \): Consider the multiplication operator \( T_a : \ell^2 \to \ell^2 \) given by
\[ T_a x = (a_j x_j)_{j \in \mathbb{N}}, \quad x = (x_j)_{j \in \mathbb{N}} \in \ell^2, \]
for some fixed \( a \in \ell^\infty \). The adjoint of \( T_a \) is the multiplication operator for the conjugate sequence \( \overline{a} \), that is \( T_a^* = T_{\overline{a}} \). Exercise: Confirm this.

iii) Multiplication operator on \( L^2[0,1] \): Consider the multiplication operator \( T_a : L^2[0,1] \to L^2[0,1] \) given by
\[ T_a f = a f, \quad f \in L^2[0,1], \]
for some fixed function \( a \in C[0,1] \). Its adjoint is the multiplication operator given by the conjugate function \( \overline{a} \), that is \( T_a^* = T_{\overline{a}} \). To see this, observe that
\[ \langle T_a f, g \rangle = \int_0^1 a(t) f(t) \overline{g(t)} \, dt = \int_0^1 \overline{f(t) a(t) g(t)} \, dt = \langle f, \overline{a}g \rangle = \langle f, T_{\overline{a}} g \rangle. \]

iv) Matrices: Consider \( \mathbb{C}^n \) with the standard inner product
\[ \langle x, y \rangle = x_1 \overline{y_1} + \ldots + x_n \overline{y_n} = x^\top \overline{y}, \]
and let \( T : \mathbb{C}^n \to \mathbb{C}^n \) be the linear map given by matrix multiplication
\[ Tx = Ax, \quad x \in \mathbb{C}^n, \]
for some fixed, \( n \times n \) matrix \( A \). Then the adjoint \( T^* \) of \( T \) is given by
\[ T^* x = \overline{A}^\top x, \quad x \in \mathbb{C}^n. \]
To see this, observe that
\[ \langle Tx, y \rangle = \langle Ax, y \rangle = (Ax)^\top \overline{y} = x^\top A^\top \overline{y} = \langle x, \overline{A}^\top y \rangle. \]

Certain classes of bounded linear operators of great practical importance can be defined by the use of adjoint operators as follows.

**Definition 1.** A bounded linear operator \( T : X \to X \) on a Hilbert space \( X \) is said to be
i) **normal** if \( T^* T = TT^* \).
ii) unitary if $T$ is bijective and $T^* = T^{-1}$. We then have
\[ T^*T = TT^* = I. \]

iii) self-adjoint or Hermitian if $T = T^*$. 

Example 4. 

i) Multiplication operator on $\ell^2$: Recall the multiplication operator $T_a$ on $\ell^2$, defined for some fixed $a \in \ell^\infty$ by
\[ T_ax = (a_jx_j)_{j \in \mathbb{N}}, \quad x \in \ell^2. \]
This is a normal operator, since it follows from $T_a^* = T_a$ that
\[ T_a^*T_a = T_aT_a^* = T_{|a|^2}. \]
We see that it is a unitary operator if and only if
\[ |a| = (|a_1|, |a_2|, |a_3|, \ldots) = (1, 1, 1, \ldots). \]
For instance, $T_a$ is unitary if
\[ a = (1, i, -1, -i, \ldots) = (i^k)_{k=0}^\infty. \]
Moreover, we see that $T_a$ is self-adjoint if and only if $a$ is real-valued, since
\[ T_a^* = T_a = T_{|a|^2}. \]

ii) Shift operator on $\ell^2$: The right shift operator $R$ on $\ell^2$ is not normal. To see this, observe that
\[ R^*Rx = L(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) =Ix, \]
but
\[ RR^*x = RLx = R(x_2, x_3, x_4, \ldots) = (0, x_2, x_3, x_4, \ldots) \neq Ix. \]

Example 5. Consider $\mathbb{R}^n$ with the standard inner product
\[ \langle x, y \rangle = x_1y_1 + \ldots + x_ny_n = x^\top y, \]
and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by matrix multiplication
\[ Tx = Ax, \quad x \in \mathbb{R}^n, \]
for some real-valued fixed, $n \times n$ matrix $A$. Then following Example 3iv), the adjoint $T^*$ of $T$ is given by
\[ T^*x = A^\top x, \quad x \in \mathbb{R}^n. \]
Consequently, we see that the matrix $A$ is
- $T$ is self-adjoint if the matrix $A$ is symmetric, meaning $A^T = A$.
- $T$ is unitary if the matrix $A$ is invertible and orthogonal, meaning $A^T = A^{-1}$.

We list certain properties of unitary operators.

Lemma 6. Let $S$ and $T$ be two unitary operators on a Hilbert space $X$. We then have:

i) $S$ is isometric; $\|Sx\| = \|x\|$ for all $x \in X$. Thus $\|S\| = 1$ for $X \neq \{0\}$. 

ii) The composition operators \( ST \) and \( TS \) are unitary.

iii) The identity operator \( I \) is unitary.

**Proof.**

i) We observe that
\[
\|Sx\|^2 = \langle Sx, Sx \rangle = \langle x, S^*Sx \rangle = \langle x, Ix \rangle = \|x\|^2.
\]

ii) We have
\[
(ST)^*(ST) = T^*S^*ST = T^*IT = T^*T = I,
\]
and by an equivalent calculation one can verify that \((ST)(ST)^* = I\).

iii) It is clear that \( I^* = I \) (i.e. the identity operator is also self-adjoint), since
\[
\langle Ix, y \rangle = \langle x, y \rangle = \langle x, Iy \rangle,
\]
for all \( x, y \in X \).

It immediately follows that \( I^*I = II^* = I \).

\[\square\]

We close our discussion of adjoint operators with certain useful relations between the kernel and range of an operator and its adjoint.

**Proposition 7.** Let \( T \) be a bounded linear operator on a Hilbert space \( X \). We then have

i) \( \overline{\text{ran}}(T) = \text{ker}(T^*)^\perp \); 

ii) \( \text{ker}(T) = \overline{\text{ran}}(T^*)^\perp \).

Equivalently, we have
\[
\overline{\text{ran}}(T^*) = \text{ker}(T)^\perp \quad \text{and} \quad \text{ker}(T^*) = \overline{\text{ran}}(T)^\perp,
\]
and consequently
\[
X = \text{ker} T \oplus \overline{\text{ran}}(T^*).
\]

**Proof.**

i) **Showing** \( \overline{\text{ran}}(T) \subseteq \text{ker}(T^*)^\perp \):

Let \( y \in \text{ran}(T) \). Then \( y = Tx \) for some \( x \in X \), and for any \( z \in \text{ker}(T^*) \), we get
\[
\langle y, z \rangle = \langle Tx, z \rangle = \langle x, T^*z \rangle = \langle x, 0 \rangle = 0.
\]
This shows that \( y \in \text{ker}(T^*)^\perp \), and thus \( \text{ran}(T) \subseteq \text{ker}(T^*)^\perp \). Finally, since \( \text{ker}(T^*)^\perp \) is closed, we must have \( \overline{\text{ran}}(T) \subseteq \text{ker}(T^*)^\perp \).

**Showing** \( \text{ker}(T^*)^\perp \subseteq \overline{\text{ran}}(T) \):

Let \( x \in \overline{\text{ran}}(T)^\perp \). Then necessarily \( x \in \text{ran}(T)^\perp \), meaning
\[
0 = \langle Ty, x \rangle = \langle y, T^*x \rangle,
\]
for all \( y \in X \).

It follows that \( T^*x = 0 \), so \( x \in \text{ker}(T^*) \). This shows \( \overline{\text{ran}}(T)^\perp \subseteq \text{ker}(T^*) \).

Taking orthogonal complements, we get
\[
\text{ker}(T^*)^\perp \subseteq \overline{\text{ran}}(T)^\perp^\perp = \overline{\text{ran}}(T).
\]

ii) **Exercise.**

\[\square\]
Corollary 8. Let $T$ be a bounded linear operator on a Hilbert space $X$. Then $\ker(T^*) = \{0\}$ if and only if $\text{ran}(T)$ is dense in $X$.

Proof. This is immediate from Proposition 7, as we have $X = \ker(T^*) \oplus \text{ran}(T)$. 

This corollary allows one to check if the range of an operator is dense in the space $X$ by determining the adjoint operator and its kernel. This can be a very useful strategy in practice, as it is often more difficult to determine the range of an operator than its kernel.