ADJOINT OPERATORS

Consider a Hilbert space X over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In this note we introduce *adjoint operators*, which provide us with an alternative description of bounded linear operators on X. We will see that the existence of so-called adjoints is guaranteed by Riesz' representation theorem.

Theorem 1 (Adjoint operator)**.** Let $T \in \mathcal{B}(X)$ be a bounded linear operator on a Hilbert space X. There exists a unique operator $T^* \in \mathcal{B}(X)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \text{for all } x, y \in X.$$

The operator T^* is called the *adjoint* of T.

Proof. Existence: Fix $y \in X$, and define the map

$$\varphi(x) = \langle Tx, y\rangle\,, \quad x \in X.$$

This is a bounded linear functional on X, as it is easily seen to be linear and

$$|\varphi(x)| = |\,\langle Tx, y\rangle\,| \leq \|Tx\|\|y\| \leq \|T\|\|y\|\|x\|.$$

By Riesz' representation theorem it follows that there exists a unique element $y^* \in X$ such that

$$\varphi(x) = \langle Tx, y\rangle = \langle x, y^*\rangle \qquad \text{for all } x \in X.$$

We thus define $T^*y := y^*$, so by definition T^* satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$. It remains to show that T^* is linear, bounded and unique.

Linearity of T^*: We have

$$\begin{aligned}\langle x, T^*(\alpha y_1 + \beta y_2)\rangle &= \langle Tx, \alpha y_1 + \beta y_2\rangle \\ &= \overline{\alpha}\langle Tx, y_1\rangle + \overline{\beta}\langle Tx, y_2\rangle \\ &= \overline{\alpha}\langle x, T^*y_1\rangle + \overline{\beta}\langle x, T^*y_2\rangle \\ &= \langle x, \alpha T^*y_1 + \beta T^*y_2\rangle \qquad \text{for all } x \in X,\end{aligned}$$

and it follows that

$$T^*(\alpha y_1 + \beta y_2) = \alpha T^*y_1 + \beta T^*y_2.$$

Boundedness of T^*: By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|T^*y\|^2 = \langle T^*y, T^*y\rangle &= \langle TT^*y, y\rangle \\ &\leq \|TT^*y\|\|y\| \\ &\leq \|T\|\|T^*y\|\|y\|.\end{aligned}$$

If $\|T^*y\| > 0$, we divide by $\|T^*y\|$ on both sides in the inequality and obtain

$$\|T^*y\| \leq \|T\|\|y\|.$$

This inequality is clearly also satisfied when $\|T^*y\| = 0$, so T^* is a bounded operator. Moreover, we have attained the additional information

$$\|T^*\| \leq \|T\|.$$

Uniqueness: Suppose there exists another operator $S \in \mathcal{B}(X)$ such that

$$\langle x, Sy \rangle = \langle x, T^*y \rangle \quad \text{for all } x, y \in X.$$

Then necessarily, for each $y \in X$, we have

$$\langle x, Sy - T^*y \rangle = 0 \quad \text{for all } x \in X.$$

It follows that $Sy = T^*y$ for every $y \in X$, meaning $S = T^*$. □

We list and prove some useful properties of adjoints.

Proposition 2. Let X be a Hilbert space, $S : X \to X$ and $T : X \to X$ be bounded linear operators and $\alpha, \beta \in \mathbb{F}$ any two scalars. We then have:

i) $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$

ii) $(ST)^* = T^* S^*$

iii) $(T^*)^* = T$

iv) $\|T^*\| = \|T\|$

v) $\|TT^*\| = \|T^*T\| = \|T\|^2$

Proof. *i*) and *ii*): *Exercise.*

iii) Fix any $y \in X$. We have

$$\begin{aligned} \langle x, T^{**}y \rangle &= \langle T^*x, y \rangle = \overline{\langle y, T^*x \rangle} \\ &= \overline{\langle Ty, x \rangle} = \langle x, Ty \rangle \end{aligned}$$

for all $x \in X$. It thus follows that $T^{**} = T$.

iv) In the proof of the existence of the adjoint, we established that $\|T^*\| \leq \|T\|$. For the opposite inequality, simply observe that by *iii*), we have

$$\|T\| = \|T^{**}\| \leq \|T^*\|,$$

and thus $\|T^*\| = \|T\|$.

v): *Exercise on Problem set 11.* □

Example 3. i) *Left and right shift operators:* Consider the right shift operator R on ℓ^2, given by

$$Rx = (0, x_1, x_2, x_3, \ldots), \quad x = (x_j)_{j \in \mathbb{N}} \in \ell^2.$$

Its adjoint is the left shift operator L, given by

$$Lx = (x_2, x_3, x_4, \ldots).$$

To see this, observe that

$$\begin{aligned}\langle Rx, y\rangle &= \langle (0, x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)\rangle \\ &= x_1\overline{y_2} + x_2\overline{y_3} + x_3\overline{y_4} + \ldots \\ &= \langle (x_1, x_2, x_3, \ldots), (y_2, y_3, y_4, \ldots)\rangle = \langle x, Ly\rangle\end{aligned}$$

for any $x, y \in \ell^2$. Thus, the operator R^* satisfying $\langle Rx, y\rangle = \langle x, R^*y\rangle$ for all $x, y \in \ell^2$ is $R^* = L$.

ii) *Multiplication operator on ℓ^2:* Consider the multiplication operator $T_a : \ell^2 \to \ell^2$ given by

$$T_a x = (a_j x_j)_{j\in\mathbb{N}}, \quad x = (x_j)_{j\in\mathbb{N}} \in \ell^2,$$

for some fixed $a \in \ell^\infty$. The adjoint of T_a is the multiplication operator for the conjugate sequence $\overline{a}$, that is

$$T_a^* = T_{\overline{a}}.$$

Exercise: Confirm this.

iii) *Multiplication operator on $L^2[0,1]$:* Consider the multiplication operator $T_a : L^2[0,1] \to L^2[0,1]$ given by

$$T_a f = af, \quad f \in L^2[0,1],$$

for some fixed function $a \in C[0,1]$. Its adjoint is the multiplication operator given by the conjugate function $\overline{a}$, that is $T_a^* = T_{\overline{a}}$. To see this, observe that

$$\langle T_a f, g\rangle = \int_0^1 a(t)f(t)\overline{g(t)}\,dt = \int_0^1 f(t)\overline{\overline{a(t)}g(t)}\,dt = \langle f, \overline{a}g\rangle = \langle f, T_{\overline{a}}g\rangle\,.$$

iv) *Matrices:* Consider $\mathbb{C}^n$ with the standard inner product

$$\langle x, y\rangle = x_1\overline{y_1} + \ldots + x_n\overline{y_n} = x^\top\overline{y},$$

and let $T : \mathbb{C}^n \to \mathbb{C}^n$ be the linear map given by matrix multiplication

$$Tx = Ax, \quad x \in \mathbb{C}^n,$$

for some fixed, $n \times n$ matrix A. Then the adjoint T^* of T is given by

$$T^*x = \overline{A}^\top x, \quad x \in \mathbb{C}^n.$$

To see this, observe that

$$\begin{aligned}\langle Tx, y\rangle &= \langle Ax, y\rangle = (Ax)^\top\overline{y} \\ &= x^\top A^\top\overline{y} = x^\top\overline{\overline{A}^\top y} = \left\langle x, \overline{A}^\top y\right\rangle.\end{aligned}$$

Certain classes of bounded linear operators of great practical importance can be defined by the use of adjoint operators as follows.

Definition 1. A bounded linear operator $T : X \to X$ on a Hilbert space X is said to be

i) *normal* if $T^*T = TT^*$.

ii) *unitary* if T is bijective and $T^* = T^{-1}$. We then have

$$T^*T = TT^* = I.$$

iii) *self-adjoint* or *Hermitian* if $T = T^*$.

Example 4. i) *Multiplication operator on ℓ^2:* Recall the multiplication operator T_a on ℓ^2, defined for some fixed $a \in \ell^\infty$ by

$$T_a x = (a_j x_j)_{j\in\mathbb{N}}, \quad x \in \ell^2.$$

This is a normal operator, since it follows from $T_a^* = T_{\overline{a}}$ that

$$T_a^* T_a = T_a T_a^* = T_{|a|^2}.$$

We see that it is a unitary operator if and only if

$$|a| = (|a_1|, |a_2|, |a_3|, \ldots) = (1, 1, 1, \ldots).$$

For instance, T_a is unitary if

$$a = (1, i, -1, -i, \ldots) = (i^k)_{k=0}^\infty.$$

Moreover, we see that T_a is self-adjoint if and only if a is real-valued, since

$$T_a^* = T_{\overline{a}} = T_a$$

only in this case.

ii) *Shift operator on ℓ^2:* The right shift operator R on ℓ^2 is *not* normal. To see this, observe that

$$R^*Rx = LRx = L(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) = Ix,$$

but

$$RR^*x = RLx = R(x_2, x_3, x_4, \ldots) = (0, x_2, x_3, x_4, \ldots) \neq Ix.$$

Example 5. Consider $\mathbb{R}^n$ with the standard inner product

$$\langle x, y\rangle = x_1y_1 + \ldots + x_ny_n = x^\top y,$$

and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by matrix multiplication

$$Tx = Ax, \quad x \in \mathbb{R}^n,$$

for some real-valued fixed, $n \times n$ matrix A. Then following Example 3iv), the adjoint T^* of T is given by

$$T^*x = A^\top x, \quad x \in \mathbb{R}^n.$$

Consequently, we see that the matrix A is

- T is self-adjoint if the matrix A is symmetric, meaning $A^T = A$.
- T is unitary if the matrix A is invertible and orthogonal, meaning $A^T = A^{-1}$.

We list certain properties of unitary operators.

Lemma 6. Let S and T be two unitary operators on a Hilbert space X. We then have:

i) S is isometric; $\|Sx\| = \|x\|$ for all $x \in X$. Thus $\|S\| = 1$ for $X \neq \{0\}$.

ii) The composition operators ST and TS are unitary.

iii) The identity operator I is unitary.

Proof. i) We observe that

$$\|Sx\|^2 = \langle Sx, Sx\rangle = \langle x, S^*Sx\rangle = \langle x, Ix\rangle = \|x\|^2.$$

ii) We have

$$(ST)^*(ST) = T^*S^*ST = T^*IT = T^*T = I,$$

and by an equivalent calculation one can verify that $(ST)(ST)^* = I$.

iii) It is clear that $I^* = I$ (i.e. the identity operator is also self-adjoint), since

$$\langle Ix, y\rangle = \langle x, y\rangle = \langle x, Iy\rangle\,, \quad \text{for all } x, y \in X.$$

It immediately follows that $I^*I = II^* = I$.

□

We close our discussion of adjoint operators with certain useful relations between the kernel and range of an operator and its adjoint.

Proposition 7. Let T be a bounded linear operator on a Hilbert space X. We then have

i) $\overline{\text{ran}(T)} = \ker(T^*)^\perp$;

ii) $\ker(T) = \text{ran}(T^*)^\perp$.

Equivalently, we have

$$\overline{\text{ran}(T^*)} = \ker(T)^\perp \quad \text{and} \quad \ker(T^*) = \text{ran}(T)^\perp,$$

and consequently

$$X = \ker T \oplus \overline{\text{ran}(T^*)}.$$

Proof. i) *Showing* $\overline{\text{ran}(T)} \subseteq \ker(T^*)^\perp$*:*
Let $y \in \text{ran}(T)$. Then $y = Tx$ for some $x \in X$, and for any $z \in \ker(T^*)$, we get

$$\langle y, z\rangle = \langle Tx, z\rangle = \langle x, T^*z\rangle = \langle x, 0\rangle = 0.$$

This shows that $y \in \ker(T^*)^\perp$, and thus $\text{ran}(T) \subseteq \ker(T^*)^\perp$. Finally, since $\ker(T^*)^\perp$ is closed, we must have $\overline{\text{ran}(T)} \subseteq \ker(T^*)^\perp$.

Showing $\ker(T^*)^\perp \subseteq \overline{\text{ran}(T)}$*:* Let $x \in \overline{\text{ran}(T)}^\perp$. Then necessarily $x \in \text{ran}(T)^\perp$, meaning

$$0 = \langle Ty, x\rangle = \langle y, T^*x\rangle\,, \quad \text{for all } y \in X.$$

It follows that $T^*x = 0$, so $x \in \ker(T^*)$. This shows $\overline{\text{ran}(T)}^\perp \subseteq \ker(T^*)$. Taking orthogonal complements, we get

$$\ker(T^*)^\perp \subseteq \overline{\text{ran}(T)}^{\perp\perp} = \overline{\text{ran}(T)}.$$

ii) *Exercise.*

□

Corollary 8. Let T be a bounded linear operator on a Hilbert space X. Then $\ker(T^*) = \{0\}$ if and only if $\operatorname{ran}(T)$ is dense in X.

Proof. This is immediate from Proposition 7, as we have

$$X = \ker(T^*) \oplus \overline{\operatorname{ran}(T)}.$$

□

This corollary allows one to check if the range of an operator is dense in the space X by determining the adjoint operator and its kernel. This can be a very useful strategy in practice, as it is often more difficult to determine the range of an operator than its kernel.