Adjoints: $T : H \to H$

$$T^* : H \to H$$

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall\ x, y \in H$$

Proposition: X Hilbert space, $S, T \in \mathcal{B}(X)$, $\alpha, \beta \in \mathbb{F}$.

i) $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$

ii) $(ST)^* = T^* S^*$

iii) $(T^*)^* = T$

iv) $\|T^*\| = \|T\|$

v) $\|T^* T\| = \|T T^*\| = \|T\|^2$

Pf iii) $(T^*)^* = T$:

$$\langle x, T^{**} y \rangle = \langle T^* x, y \rangle$$

$$= \overline{\langle y, T^* x \rangle}$$

$$= \overline{\langle Ty, x \rangle} = \langle x, Ty \rangle$$

$\forall\ x, y \in X$

$$\Rightarrow \quad T^{**} y - Ty = 0 \quad \forall\ y \in X$$

$$\Rightarrow \quad T^{**} = T$$

iv) Know already that $\|T^*\| \leq \|T\|$

On the other hand, $T^{**} = T$, so

$$\|T\| = \|T^{**}\| \leq \|T^*\|$$

$$\Rightarrow \|T\| = \|T^*\| .$$

∎

Exs:

i) Left and right shift operators on ℓ^2:

$R : \ell^2 \to \ell^2$ given by $R(x) = (0, x_1, x_2, \dots)$

$(x = (x_1, x_2, x_3, \dots) \in \ell^2)$

Its adjoint is

$L : \ell^2 \to \ell^2$ given by $L(x) = (x_2, x_3, x_4, \dots)$

Why:

$$\langle Rx, y \rangle = \left(\langle x, \overset{\overset{L\,?}{\parallel}}{R^*} y \rangle \right)$$

$$= \langle (0, x_1, x_2, \dots), (y_1, y_2, y_3, \dots) \rangle$$

$$= \cancel{0 \cdot \overline{y_1}} + x_1 \cdot \overline{y_2} + x_2 \overline{y_3} + \dots$$

$$= \langle (x_1, x_2, x_3, \dots), \underbrace{(y_2, y_3, y_4, \dots)}_{Ly} \rangle$$

$$= \langle x, Ly \rangle \quad \forall\, x, y \in \ell^2$$

$$\Rightarrow R^* = L .$$

ii) Mult. operator: Consider $T_a : \ell^2 \to \ell^2$ given by $T_a x = (a_j x_j)_{j \in \mathbb{N}}$, $x \in \ell^2$, where $a = (a_j) \in \ell^\infty$. What is T_a^*?

<u>Claim</u>: $T_a^* = T_{\bar{a}}$

<u>Exercise</u>: Show this! $(\langle T_a x, y \rangle = \langle x, T_{\bar{a}} y \rangle)$

iii) Mult. operator on $L^2[0,1]$: Consider $T_a : L^2[0,1] \to L^2[0,1]$ given by

$$T_a(f) = af, \quad f \in L^2[0,1]$$

$T_a f(t) = a(t) f(t)$

for some fixed fnc $a \in C[0,1]$. We have $T_a^* = T_{\bar{a}}$:

$$\begin{aligned}
\langle T_a f, g \rangle &= \int_0^1 T_a f(t) \overline{g(t)} \, dt \\
&= \int_0^1 a(t) f(t) \overline{g(t)} \, dt \\
&= \int_0^1 f(t) \overline{\overline{a(t)} g(t)} \, dt \\
&= \langle f, T_{\bar{a}} g \rangle \quad \forall f, g \in L^2[0,1].
\end{aligned}$$

iv) Matrices: Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be given by $Tx = Ax$ for fixed $n \times n$ matrix A. Then

$$T^* x = \overline{A}^T x \,, \qquad x \in \mathbb{C}^n$$

Why: $\langle Tx, y \rangle = \langle Ax, y \rangle = (Ax)^T \overline{y}$

$$= x^T A^T \overline{y} = x^T (\overline{\overline{A}^T y})$$

$$= \langle x, \overline{A}^T y \rangle \quad \forall\, x, y \in \mathbb{C}^n$$

$\Rightarrow T^*$ is given by $T^* x = \overline{A}^T x$.

DEF: X Hilbert space, $T \in B(X)$. We say that T is:

i) normal if $TT^* = T^*T$

ii) unitary if T is a bijection and $T^{-1} = T^*$. We then have $TT^* = T^*T = I$

iii) self-adjoint (or Hermitian) if $T^* = T$

Exs: i) Mult. op. on ℓ^2: $T_a : \ell^2 \to \ell^2$

$$T_a(x) = \underset{\ell^\infty}{\underset{\uparrow}{a}}\, x$$

We claimed: $T_a^* = T_{\overline{a}}$

T_a is normal:

$$\left.\begin{aligned} T_a^* T_a &= T_{\overline{a}} T_a = T_{|a|^2} \\ T_a T_a^* &= T_a T_{\overline{a}} = T_{|a|^2} \end{aligned}\right\} = I \,?$$

We see that T_a is unitary iff:

$$|a|^2 = (|a_1|^2, |a_2|^2, |a_3|^2, \dots) = (1, 1, 1, \dots)$$

$\Rightarrow$ T_a is unitary if $|a_i| = 1 \ \forall i$

For instance, T_a is unitary if

$$a = (i^k)_{k=0}^{\infty} = (1, i, -1, -i, 1, i, \dots)$$

Finally, T_a is self-adj iff a is real-valued: $T_a^* = T_{\bar{a}} = T_a \iff a_j \in \mathbb{R} \ \forall j$

ii) Shift operators on ℓ^2: $R: \ell^2 \to \ell^2$ is not normal, as:

$$R^* R x = L R x = L((0, x_1, x_2, x_3, \dots)) = (x_1, x_2, \dots) = Ix$$

$$R R^* x = R L x = R((x_2, x_3, x_4, \dots)) = (\underline{0}, x_2, x_3, x_4, \dots) \neq Ix$$

Ex: $T: \mathbb{C}^n \xrightarrow{\mathbb{R}^n \to \mathbb{R}^n} \mathbb{C}^n$ given by $Tx = Ax \ \forall x \in \mathbb{C}^n$

Saw: $T^* x = \bar{A}^T x \ \forall x \in \mathbb{C}^n$

Thus if $T: \mathbb{R}^n \to \mathbb{R}^n$ is given by real-val.

$(n \times n)$ matrix A, then $T^* x = A^T x \quad \forall\, x \in \mathbb{R}^n$

We see that:

i) T is self-adj. if A is symmetric $(A = A^T)$

ii) T is unitary if A is invertible and $A^{-1} = A^T$ (columns of A are orthon.).

<u>Lemma</u>: X Hilbert space, $S, T \in \mathcal{B}(X)$ unitary.

Then: i) S is isometric: $\|Sx\| = \|x\| \quad \forall\, x \in X$

$$\Rightarrow \|S\| = 1 \quad (X \neq \{0\})$$

ii) ST and TS are unitary.

iii) I (the identity operator) is unitary.

<u>Pf</u> i):

$$\|Sx\|^2 = \langle Sx, Sx \rangle = \langle x, \underbrace{S^* S}_{= S^{-1} S = I} x \rangle$$

$$= \langle x, Ix \rangle = \langle x, x \rangle = \|x\|^2$$

$$\Rightarrow \|Sx\| = \|x\| \quad \forall\, x \in X.$$

<u>Proposition</u>: X Hilbert space, $T \in \mathcal{B}(X)$. We then have:

i) $\overline{\operatorname{ran}}(T) = (\ker T^*)^\perp$

$\overline{\operatorname{ran}(T^*)} = (\ker T)^\perp$

ii) $\ker(T) = \operatorname{ran}(T^*)^\perp$

$\ker(T^*) = \operatorname{ran}(T)^\perp$

Consequently, range = ran

$$X = \ker(T) \oplus \overline{\operatorname{ran}}(T^*)$$

$\ker(T^*) \oplus \overline{\operatorname{ran}}(T)$

meaning every $x \in X$ has a unique repr.

$$x = x_{\ker(T)} + x_{\ker(T)^\perp}$$

$$= x_{\ker T} + x_{\overline{\operatorname{ran}}(T^*)}$$

<u>Corollary</u>: X Hilbert space, $T \in \mathcal{B}(X)$

Then $\ker(T^*) = \{0\}$ <u>iff</u> $\operatorname{ran}(T)$ is dense in X. (i.e. if $\overline{\operatorname{ran}}(T) = X$)

<u>Proof of i)</u>: $\overline{\operatorname{ran}}(T) = \ker(T^*)^\perp$

Showing $\overline{\operatorname{ran}}(T) \subseteq \ker(T^*)^\perp$

Show: $\operatorname{ran}(T) \subseteq \ker(T^*)^\perp \leftarrow$ closed, so it will follow that $\overline{\operatorname{ran}}(T) \subseteq \ker(T^*)^\perp$

Let $y \in \operatorname{ran}(T) \rightarrow y = Tx$ for some $x \in X$

For any $z \in \ker(T^*)$, we have

$$\langle y, z\rangle = \langle Tx, z\rangle = \langle x, \underbrace{T^* z}_{=0}\rangle$$

$$= \langle x, 0\rangle = 0$$

$\Rightarrow y \in \ker(T^*)^\perp \Rightarrow \operatorname{ran}(T) \subseteq \ker(T^*)^\perp$

Showing $\ker(T^*)^\perp \subseteq \overline{\operatorname{ran}}(T)$

Show: $\overline{\operatorname{ran}}(T)^\perp \subseteq \ker(T^*)$

Then $\overline{\operatorname{ran}}(T)^{\perp\perp} = \overline{\operatorname{ran}}(T) \supseteq \ker(T^*)^\perp$

Let $x \in \overline{\operatorname{ran}}(T)^\perp$. Then $x \in \operatorname{ran}(T)^\perp$

$$\Rightarrow 0 = \langle Ty, x \rangle = \langle y, \underbrace{T^*x}_{=0} \rangle \quad \forall\, y \in X$$

$$\Rightarrow T^*x = 0, \text{ so } x \in \ker(T^*)$$

$$\Rightarrow \overline{\operatorname{ran}}(T)^\perp \subseteq \ker(T^*)$$

$$\Rightarrow \overline{\operatorname{ran}}(T) \supseteq \ker(T^*)^\perp$$

$\blacksquare$

SELECTED TOPICS IN LINEAR ALGEBRA

$\mathcal{B}(X, Y)$ with X and Y finite-dimensional

1) Showing $\mathcal{B}(X,Y) \cong \mathcal{M}_{m \times n}(\mathbb{C})$ ($m \times n$ matrices with complex entries) if X and Y are complex vector spaces of dim. n and m, respectively.

2) Spectral theory for such operators

3) Matrix decompositoons (SVDs, polar decomp, pseudoinverses)

1) Bdd lin operators between finite-dim spaces

<u>THM</u>: Let X be a complex vector space with basis $\{e_1, \ldots, e_n\}$. Then X is isomorphic to $\mathbb{C}^n$, $X \cong \mathbb{C}^n$ (meaning $\exists$ linear bijection between the two). Similarly, if X is a real vector space, then $X \cong \mathbb{R}^n$.

Sketch
<u>Pf</u> (complex case):

Every $x \in X$ has a unique repr

$$x = \sum_{i=1}^{n} a_i e_i \quad , \quad a_i \in \mathbb{C}$$

Let $T : X \to \mathbb{C}^n$ be defined by

$$Tx = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$$

Then T is linear, injective and surjective (check this!), so $X \cong \mathbb{C}^n$ ∎

Now let $T : X \to Y$

fin-dim

Lemma:

Let X be fin-dim vector space with basis $\{e_1, \dots, e_n\}$. For any values $y_1, \dots, y_n \in Y$ there exists precisely one $T \in B(X,Y)$ s.t.

$$Te_j = y_j \quad \forall \; j = 1, \dots, n.$$

Proof: In note.

$$x = \sum_{i=1}^{n} a_i e_i \longrightarrow Tx = \sum_{i=1}^{n} a_i Te_i$$
$$= \sum_{i=1}^{n} a_i y_i$$

Ex: $T : \mathbb{C}^n \to \mathbb{C}^m$

$Tx = Ax$, $A \in \mathcal{M}_{m \times n}(\mathbb{C})$

Applying T to the std basis vectors determines the column vectors A_j of A

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$