CURRICULUM REVIEW

CH 1:
- Sets, subsets, basic set operations
- Functions and basic notations
- Cardinality: finite, countably infinite, uncountable

$f : \{1, \dots, n\} \to X$, $f : \mathbb{N} \to X$

bijections

- Schröder-Bernstein Thm:

$$\left.\begin{array}{l} X \xrightarrow{\text{inj}} Y \\ Y \xrightarrow{\text{inj}} X \end{array}\right\} \Rightarrow \text{Exists bijection: } X \to Y$$

- $\underbrace{Y \subseteq X}_{\text{countable}} \Rightarrow Y$ countable

$\underbrace{X \subseteq Y}_{\text{uncount.}} \Rightarrow Y$ uncountable

- Suprema and infima
  - Suprema: least upper bound
  - Infima: greatest lower bound

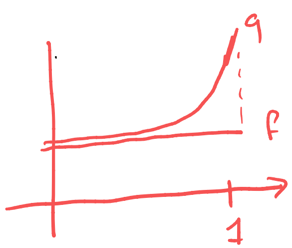
- Note: Any bounded set in $\mathbb{R}$ <u>has</u> a supremum and an infimum.

- Ch 1.7-1.9: Self-study

- Vector spaces:
  - over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$
  - vector space axioms
  - vector subspace
  - span of $A \subseteq V$
  - (finitely) linear independent set $A \subseteq V$
  - Hamel basis $\mathcal{B}$ of V : lin. indep + $\text{span}(\mathcal{B}) = V$
  - $\text{dimension}(V) = d$ if cardinality of $\mathcal{B}$ is d
    $= \infty$ if $\mathcal{B}$ is an inf Hamel basis
  - given Hamel basis $\mathcal{B} = \{x_1, \dots, x_d\}$ for V, every $x \in V$ has a <u>unique</u> repr. $x = \sum_1^d c_i x_i$

CH 2: Metric Spaces

- Metrics: - metric axioms



$d(f,g)$ "small" if

$$d(f,g) = \int_0^1 |f-g| \, dt$$

and "large" if

$$d(f,g) = \sup_{0 \le t \le 1} |f(t) - g(t)|$$

- Convergence in (X, d): $(x_n)_{n\in\mathbb{N}}$, $x_n \to x \in X$ if $\forall\, \varepsilon > 0\ \exists\, N$ s.t $d(x, x_n) < \varepsilon\ \forall\, n > N$.
- Cauchy seq. in (X, d): $(x_n)_{n\in\mathbb{N}}$ satisfies $\forall\, \varepsilon > 0\ \exists\, N$ s.t. $d(x_n, x_m) < \varepsilon\ \forall\, n, m > N$
- Convergence $\Rightarrow$ Cauchy
  $\not\Leftarrow$?
- Completeness: Every Cauchy sequence converges.
- $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ are complete metric spaces.
- $(\ell^1, \|\cdot\|_1)$ and $(\ell^\infty, \|\cdot\|_\infty)$ are complete:
  (x_i) Cauchy : i) Find candidate for the limit x
  ii) Show x is in the space
  iii) Show $x_i \to x$ in appropriate metric
- $(C[0,1], \|\cdot\|_1)$ is not complete.
- Open balls $B_r(x) = \{ y \in X : d(x,y) < r \}$
- Bounded, open and closed sets
- Interior and closure of sets

- Saw: $E \subseteq X$ is closed

$$\Updownarrow$$

$$(x_n) \subseteq E, \; x_n \to x \in X \implies x \in E$$

- Saw: $E \subseteq X$ closed and (X, d) is complete

$$\implies (E, d) \text{ is complete.}$$

- Boundary points of a set
- Dense subset $E \subseteq X$: $\overline{E} = X$
- Separability : Countable dense subset
- Compactness (new!) : Every open cover $\{E_i\}$ of $K \subseteq X$ has a finite subcover.
    - Saw: $K \subseteq X$ compact $\implies$ K bounded and closed
- Continuity of fncs between metric spaces:
    i) Cont. $f: X \to Y$:
        1) Inv image of open sets are open
        2) $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$
        3) $x_n \to x$ in $X \implies f(x_n) \to f(x)$ in Y

    ii) Uniform cont.

iii) Lipschitz cont.

Note 1: BFPT

- Fixed point of $T: X \to X$: $Tx = x$
- Contraction: $d(Tx, Ty) \leq K d(x,y)$ for some $K < 1$.
- BFPT: Contraction T on a complete space $\Rightarrow$ T has a unique fixed point.
- Error bounds for the iteration $x_0 \in X$, $x_{n+1} = Tx_n$: i) $d(x_m, x) \leq \frac{K^m}{1-K} d(x_0, x_1)$

  ii) $d(x_m, x) \leq \frac{K}{1-K} d(x_{m-1}, x_m)$
- Applications

CH 3: Norms and Banach Spaces

- Norm axioms
- Normed spaces $(X, \|\cdot\|)$ vector space!
- ℓ^p spaces: $x \in \ell^p$, $\|x\|_p^p = \sum_{i=1}^{\infty} |x_i|^p$ $p < \infty$

  $\|x\|_\infty = \sup_i |x_i|$

- Saw: $1 \leq p < q \leq \infty \Rightarrow \ell^p \subsetneq \ell^q$
- Hölder's inequality: $\|xy\|_1 \leq \|x\|_p \|y\|_q$
- Minkowski's ineq: $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

  $\Rightarrow (\ell^p, \|\cdot\|_p)$ is a normed space.
- Norms induce metrics: $d(x,y) = \|x-y\|$
- $h_{xy} = \{tx + (1-t)y : 0 \leq t \leq 1\}$:

  line segment joining x and y
- Convex sets: $K \subseteq X$ convex if given $x, y \in K$ we have $h_{xy} \subseteq K$
- Open balls in $(X, \|\cdot\|)$ are convex
- Banach space = complete normed spaces

  Exs: i) $(\ell^p, \|\cdot\|_p)$ for any $1 \leq p \leq \infty$

  ii) $(\mathbb{F}^p, \|\cdot\|_p)$ — " —

  iii) $(Y, \|\cdot\|)$ for any closed $Y \subseteq X$ where $(X, \|\cdot\|)$ is Banach

  iv) $(C[a,b], \|\cdot\|_\infty)$ (but not for any $p < \infty$)

  $$\Big(\underbrace{(C[a,b], \|\cdot\|_p)}_{\text{dense}} \subseteq (\underset{\substack{\uparrow \\ \text{complete}}}{L^p[a,b]}, \|\cdot\|_p)\Big)$$

- Equivalent norms:
  i) def: $c_1 \|x\|_a \leq \|x\|_b \leq c_2 \|x\|_a$
  ii) showing norms are equiv / not equiv.
  iii) finite-dim space: all norms are equiv.

CH 4: Further results on Banach Spaces

- Convergence of series in normed spaces:

$$\sum_{i=1}^{\infty} x_i \to x \quad \text{if} \quad S_N = \sum_{i=1}^{N} x_i \to x$$

as $N \to \infty$

- Span, closed span, complete sequence

  all conv $\sum c_n x_n$

  ↑ not to be mixed with complete space

- Hamel bases and Schauder bases $\mathcal{B}$ of X

  ↑ lin indep, $\text{span}(\mathcal{B}) = X$

  ↑ Every element $x \in X$ has unique series repr $x = \sum c_n(x) b_n$, $b_n \in \mathcal{B}$

- Weierstraß Approx. Thm:

  Poly. version:
  $f \in C[a,b]$; $\varepsilon > 0$ $\exists$ p_N s.t. $\|f - p_N\|_\infty < \varepsilon$

  (Trig. version)

CH 5: Inner Products and Hilbert Spaces

- inner product axioms
- induces a norm → satisfies the parallelogram law
- Cauchy-Schwarz ineq.
  Used to show $\|x\|^2 = \langle x, x \rangle$ is a norm
- Hilbert space: Cauchy sequences converge (completeness)

- Exs: i) $\mathbb{R}^d$ and $\mathbb{C}^d$ with normal dot prod
  ii) ℓ^2 (but no other ℓ^p) with
  $$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$
  iii) $L^2[a,b]$ ($\supseteq C[a,b]$ dense)
  with
  $$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)}\, dt$$
  Leb. int.

- Orthogonality of elements, seqs, sets.
- Orthogonal complements:
  $$A^\perp = \{x \in H : x \perp A\}$$
  Properties of ortho comp:
  Exs: $A \subseteq (A^\perp)^\perp$
  $A^\perp \subseteq H$ closed subspace

- Closest Point Theorem : - distance
  - closest point

- Orthogonal Projection and Projection Thm.

- Consequence : Span of vectors $\{x_n\}$ in H is dense iff $(\text{span}\{x_n\})^\perp = \{0\}$.

- Orthonormal sequences and bases :

  i) ONS $\{e_n\}$: - Bessel's ineq
  - $\sum \langle x, e_n \rangle e_n$ ortho proj of x onto $\text{span}\{e_n\}$
  - conv $\sum c_n e_n$

  ⋮

  ii) ONB $\{e_n\}$: - Plancherel's and Parseval's equalities
  - completeness
  - $x = \sum \langle x, e_n \rangle e_n$ uniquely

Note:

$$\left.\begin{array}{l}\text{Orthornormality} \\ + \text{completeness}\end{array}\right\} \Longrightarrow \begin{array}{l}\text{ONB, Unique} \\ \text{representation} \\ x = \sum \langle x, e_n \rangle e_n\end{array}$$

Lin. indep + completeness $\Big\} \not\Rightarrow$ Schauder, i.e. unique repr. $x = \sum c_n(x) e_n$

- Minimality of $\{x_n\} \subseteq H$: $x_m \notin \overline{\operatorname{span}}\{x_n\}_{n \neq m}$
- ONB for all separable Hilbert spaces
- Gram-Schmidt orthogonalization
- The complex trigonometric system $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}} = \{e^{2\pi i n t} : n \in \mathbb{Z}\}$ is an ONB for $L^2[0,1]$.

CH 6: Operator Theory

- Linear operators between normed spaces:
  - i) linearity
  - ii) functionals $\mu : X \to \mathbb{F}$
  - iii) boundedness: $\|Tx\| \leq C\|x\|$
  - iv) $\|T\| = \sup\limits_{\|x\|=1} \|Tx\| = \sup\limits_{x\neq 0} \dfrac{\|Tx\|}{\|x\|}$
- $T: X \to Y$ linear and X finite-dim $\Rightarrow T$ bdd.
- Examples

- Bddness = continuity for linear operators
- $\mathcal{B}(X,Y) = \{ A : X \to Y : A \text{ bdd and linear} \}$

Showed $(\mathcal{B}(X,Y), \|\cdot\|)$ is a normed space

op. norm

- THM: When X is normed and Y is Banach, then $\mathcal{B}(X,Y)$ is Banach.
- Composition of operators : $\|BA\| \le \|B\| \cdot \|A\|$
- Isometries :

not spec. in book

i) between vector spaces : bdd, lin bijection

ii) — " — normed spaces: — " —

and $m\|x\| \le \|Tx\| \le M\|x\|$

- Isometric Isomorphisms : $m = M = 1$ in
- $T : H \to \ell^2$ def by $x \to \{ \langle x, e_n \rangle \}$

separable

ONB of $\mathcal{H}$

is an isometric isomorphism.

$\Rightarrow H \cong \ell^2$

- Dual spaces : $X^* = \mathcal{B}(X, \mathbb{F})$, always complete!

bc $\mathbb{F}$ is complete

- Riesz repr thm: $H \cong H^*$

  $\mu \in H^* \Rightarrow \mu(x) = \langle x, y \rangle$ for some fixed $y \in H$, and $\|\mu\| = \|y\|$

Note 2: Adjoint Operators

- $T \in \mathcal{B}(X)$, T^* : $\langle Tx, y \rangle = \langle x, T^* y \rangle \ \forall x, y \in X$

  └ Hilbert space

- Properties of these
- Examples
- Normal, unitary and self-adjoint operators

  $TT^* = T^*T$ $\quad T^* = T^{-1}$ $\quad T^* = T$

- T repr by A $\Rightarrow$ T^* repr by $\bar{A}^T = A^*$

- i) $\overline{\mathrm{ran}(T)} = \ker(T^*)^\perp$

  ii) $\ker(T) = \mathrm{ran}(T^*)^\perp$

  $\Rightarrow \ker(T^*) = \{0\}$ iff $\mathrm{ran}(T)$ is dense

Note 3: Topics in Linear Algebra

- $\mathcal{B}(X, Y) \cong \mathcal{M}_{m \times n}(\mathbb{C})$

  finite-dim

- Eigenvalues and eigenvectors
- Diagonalization of matrices $S^{-1}AS = \Lambda$
- Props of e.values and e.vectors for unitary and self-adj matrices.
- Similar matrices : $A = M^{-1}BM$
- Schur's triangulation lemma: Can always attain triangular $U^*AU = T$ ← triangular

consequence

- Spectral thm for Hermitian matrices:

  A self-adj $\Rightarrow$ $A = U\Lambda U^*$

  └ diagonal
- Spectral thm:

  A normal $\iff$ $A = U\Lambda U^*$

  └ diag.
- Singular Value Decomp (SVD):

  i) Positive-definite matrices (self-adj)

  ii) Singular values of A: Sq. root of pos. e.values of AA^* (or equiv A^*A).

  iii) $A = U\Sigma V^*$ (with "recipe")

  iv) $\|A\| = \sigma_1$ (largest sing. value)

  v) Polar decomp. for square A:

$$A = \underbrace{UV^*}_{\text{unitary}} \underbrace{V\Sigma V^*}_{\text{pos semi-def}} = WP$$

vi) Pseudoinverse : $A^+ = V\Sigma^+ U^*$

$Ax = b$: consistent: A^+b is the unique min. norm solution

inconsistent: A^+b is best approx to a sol with minimal norm.