5: Singular Value Decomp

<u>DEF</u>: A self-adjoint $A \in M_{n\times n}(\mathbb{C})$ is <u>positive definite</u> if

$$\langle Ax, x\rangle > 0 \quad \forall \text{ nonzero } x \in \mathbb{C}^n$$

If: $\langle Ax, x\rangle \geq 0 \ \forall \ x \in \mathbb{C}^n$, then A is <u>positive semi-definite</u>.

<u>Prop</u>: Self-adj $A \in M_{n\times n}(\mathbb{C})$ is pos. def <u>iff</u> all eigenvalues of A are positive. Similarly, A is pos. semi-def. <u>iff</u> all e.values are nonneg.

<u>Proof</u>: $\Rightarrow$) Say A is pos. def.

$$\langle Ax, x\rangle > 0 \quad \forall \ x \in \mathbb{C}^n \setminus \{0\}$$

In particular,

$$0 < \langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda \underbrace{\|v\|^2}_{>0}$$

$$\Rightarrow \lambda > 0$$

$\Leftarrow$) Suppose all e.values of A are positive. By the Spectral Thm,

$$A = U^* \Lambda U \leftarrow \text{unitary}$$

$\uparrow$ pos. e.values on diagonal

Thus,

$$\langle Ax, x \rangle = \langle U^*\Lambda U x, x \rangle$$
$$= \langle \Lambda \underbrace{Ux}_{y}, Ux \rangle = \langle \Lambda y, y \rangle$$

for $y = Ux \in \mathbb{C}^n$. Note that

$$\langle \Lambda y, y \rangle = \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \ldots + \lambda_n |y_n|^2 > 0$$

for nonzero $y \in \mathbb{C}^n$. Finally, note that $y = Ux = 0$ <u>iff</u> $x = 0$. ∎

We are interested in the pair A^*A and AA^*

<u>Corollary</u>: Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Then the $(n\times n)$ matrix A^*A and $(m\times m)$ matrix AA^* are self-adj. with nonneg. eigenvalues, and the positive e.values coincide.

Note: $m \neq n$, $m >> n$

$\uparrow$ at most n nonzero e.vals.

<u>Lemma</u>: For any $A \in M_{m \times n}(\mathbb{C})$ and $B \in M_{n \times m}(\mathbb{C})$, the matrices AB and BA have the same nonzero eigenvalues

<u>Proof</u>: Prob. Set 13.

<u>Proof Cor.</u>:

Clear that AA^* and A^*A are self-adj.

$0 \leq \|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle \rightarrow$ pos. semi-def.

$\Rightarrow$ E.values of A^*A are non.-neg.

Similarly: $0 \leq \|A^*x\|^2 = \ldots = \langle AA^*x, x \rangle$
$\rightarrow$ pos. semi-def.

By Lemma above,
the positive e.values of AA^* and A^*A coincide. $\blacksquare$

<u>DEF</u>: Let $A \in M_{m \times n}(\mathbb{C})$ have rank r. Let $\sigma_1^2 \geq \ldots \geq \sigma_r^2$ be the positive eigenvalues of A^*A (or AA^*). The scalars $\sigma_1, \ldots, \sigma_r$ are called the <u>positive singular values</u> of A.

<u>THM (SVD)</u>

Suppose $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ is of rank r, and let $\sigma_1 \geq \dots \geq \sigma_r$ be the positive singular values of A. Let Σ be the $(m\times n)$ matrix defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i=j \leq r \\ 0 & \text{otherwise} \end{cases}$$

Then there exists an $(m\times m)$ unitary matrix U and an $(n\times n)$ unitary matrix V s.t.

$$A = U \Sigma V^*$$

<u>Proof</u>: A^*A self-adj and $(n\times n)$ with pos. e.values $\sigma_1^2 \geq \dots \geq \sigma_r^2$ and $(n-r)$ zero e.values

By the Spectral Theorem $\exists$ $(n\times n)$ unitary V s.t. $\circledast$

$$V^*A^*AV = (AV)^*(AV) = D$$

where $D = \Sigma^*\Sigma$ is the $(n\times n)$ diag. matrix with

$$D_{ii} = \sigma_i^2 \quad , \quad i = 1, \dots, r$$

and zeros elsewhere.

Recall: From ⊛ we have

$$V^*A^*AV = V^{-1}A^*AV = D$$
$$A^*AV = VD$$

$\Rightarrow$ Columns of V are the (orthon.) e.vectors of A^*A.

Now note: from ⊛ clear that columns $(AV)_j$ of AV are orthogonal and length of $(AV)_j$ is σ_j (bc $\langle (AV)_j, (AV)_j \rangle = \|(AV)_j\|^2 = \sigma_j^2$) for $j = 1, \ldots, r$.

Let U_r denote the $(m \times r)$ matrix with $(AV)_j / \sigma_j$ as its j^{th} column:

$$U_r = \begin{bmatrix} | & & | \\ \frac{(AV)_1}{\sigma_1} & \cdots & \frac{(AV)_r}{\sigma_r} \\ | & & | \end{bmatrix}$$

Now complete U_r to an $(m \times m)$ unitary matrix U by finding an ONB to the ortho. comp. of the column space of U_r:

$$U = \left[\; U_r \; \vdots \; \text{ONB} \; \right]$$

Use these $(m-r)$ basis vectors as the last columns of U. We then have

$$AV = U\Sigma \iff A = U\Sigma V^*$$

$\blacksquare$

<u>Rmks</u>: • The last $(m-r)$ columns of U and the last $(n-r)$ columns of V are redundant. As a consequence, the SVD of A is not uniquely determined.

<u>Ex</u>: $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$$A^*A = \begin{bmatrix} 3 & 3 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$AA^* = \dots = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$$\det(AA^* - \lambda I) = (17-\lambda)^2 - 8^2 = (\lambda-9)(\lambda-25) = 0$$

A^*A

$\lambda = 9$ or $\lambda = 25$

$\sigma_1^2 = 25 \quad (\sigma_1 = 5)$

$\sigma_2^2 = 9 \quad (\sigma_2 = 3)$

$\sigma_3^3 = 0 \leftarrow$ Last eigenvalue of <u>A^*A</u>

$\sigma_1^2 = 25$:

$$A^*A - \sigma_1^2 I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 17/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\text{solve}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightarrow v_1 = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$ for some $t \in \mathbb{R}\setminus\{0\}$,

and a normalized choice is e.g. $v_1 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$

$\sigma_2^2 = 9$:

$$A^*A - \sigma_2^2 I = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \overset{\text{solve}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightarrow v_2 = \begin{pmatrix} t/4 \\ -t/4 \\ t \end{pmatrix}$ for some $t \in \mathbb{R}\setminus\{0\}$

and a normalized choice is e.g.

$$v_2 = \left(\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)^T$$

$\sigma_3^2 = 0$:

$$A^*A - \sigma_3^2 I = A^*A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \overset{\text{solve}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightarrow v_3 = (-2t, 2t, t)^T$, and normalized choice is

$$v_3 = \left(\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{1}{3}\right)^T.$$

We can now "build" all matrices that we need:

I $\quad V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/6 & 2/3 \\ \sqrt{2}/2 & -\sqrt{2}/6 & -2/3 \\ 0 & 2\sqrt{2}/3 & -1/3 \end{bmatrix}$

II $\quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

III $\quad U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \frac{Av_1}{\sigma_1} & \frac{Av_2}{\sigma_2} \\ | & | \end{bmatrix}$

$$Av_1 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2}/2 \\ 5\sqrt{2}/2 \end{bmatrix} \rightarrow \frac{Av_1}{\sigma_1} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$\sigma_1 = 5$

$$Av_2 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ 2\sqrt{2}/3 \end{bmatrix} = \begin{bmatrix} 9\sqrt{2}/6 \\ -9\sqrt{2}/6 \end{bmatrix} \rightarrow \frac{Av_2}{\sigma_2} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$\sigma_2 = 3$

$$\Rightarrow U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = U \Sigma V^*,$$ with I, II, III

<u>Proposition</u>: Let $A \in M_{m\times n}(\mathbb{C})$ have pos. singular values $\sigma_1 \geq \dots \geq \sigma_r$. The operator norm of A is then

$$\|A\| = \sigma_1 .$$

<u>Pf</u>: $\|A\| = \sup_{\|x\|=1} \|Ax\|$

Let $A = U\Sigma V^*$ (the SVD), and let v_1 be the first column of V. Then $\|v_1\| = 1$ (unitary V) and since

$$AV = U\Sigma$$

it is clear that $Av_1 = \sigma_1 \underbrace{u_1}_{\|u_1\|=1} \Rightarrow \|Av_1\| = \sigma_1$

$$\Rightarrow \|A\| = \sup_{\|x\|=1} \|Ax\| \geq \|Av_1\| = \sigma_1$$

Now let $x \in \mathbb{C}^n$ be any vector with $\|x\| = 1$.

Consider $Ax = U\Sigma \underbrace{V^*x}_{y,\ \|y\|=1}$

V^* unitary $\to$ repr. isometry $\to \|V^*x\| = \|x\| = 1$

Note: $\Sigma y = \begin{pmatrix} \sigma_1 y_1 \\ \sigma_2 y_2 \\ \vdots \\ \sigma_r y_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \|\Sigma y\| \leq \sigma_1 \|y\|$

Finally, since U is also unitary:

$$\|Ax\| = \|U\Sigma y\| = \|\Sigma y\| \leq \sigma_1 \|y\| = \sigma_1$$

$$\Rightarrow \|A\| = \sup_{\|x\|=1} \|Ax\| = \sigma_1$$

$\blacksquare$

$z = x + iy = |z| e^{i\theta}$, $|e^{i\theta}| = 1$

↑
nonneg

Square matrix A: $A = WP$

unitary ($\sim e^{i\theta}$) ↗ ↖ positive semi-def ($\sim r = |z|$)

<u>THM:</u> For any $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ $\exists$ unitary W and pos. semi-def P s.t.

$$A = WP$$

<u>Proof</u>: By the SVD thm:

$$A = U\Sigma V^* = UV^*V\Sigma V^*$$
$$= WP$$

with $W = UV^*$ and $P = V\Sigma V^*$