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5.8: orthonormal bases

Definition:

A countably infinite orthonormal seq. $\{e_n\}$ that is complete in a Hilbert space H is called an orthonormal basis.

Complete orthonormal seq. $\Leftrightarrow$ Orthonormal basis!

Every x can be written:

$$x = \sum \langle x, e_n \rangle e_n$$

$$\Downarrow$$

Schauder basis:

complete and linearly independant seq. $\not\longrightarrow$ linearly independance and completeness.

Recall:

Weierstrass approximation:

Say, given any $f \in C[0,1]$ and any $\varepsilon > 0$, we can find:

$$P_N = \sum_{n=0}^{N} c_n t^n \quad \text{s.t:} \quad \|f - P\|_u < \varepsilon$$

$\longrightarrow$ Monomials $\{t^n\}_{n=0}^{\infty}$ are complete in $(C[0,1], \|\cdot\|_u)$, but this does not imply that every $f \in C[0,1]$ can be written:

$$f = \sum_{n=0}^{\infty} c_n t^n$$

in an unique way.

Definition: Minimality

Let $\{x_n\} \in H$ (Hilbert space).

This seq. is minimal if $x \notin \overline{\operatorname{span}}\{x_n\}_{n \neq m}$

In a Hilbert space:

$\{x_n\}$ is a ONB $\Rightarrow$ Schauder basis

$\Rightarrow$ Minimal

$\Rightarrow$ Linearly independent.

Ex:

i) $\{\delta_n\}$ is an ONB in ℓ^2

Given any $x = (x_1, x_2, x_3, \dots) \in \ell^2$

$\langle x, \delta_n \rangle = x_k \Rightarrow x = \sum \langle x, e_k \rangle e_k$

ii) If $\{e_1, \dots, e_d\}$ is orthon. and complete in a Hilbert space H (of dim d), then $\{e_n\}_{n=1}$ is an ONB.

Ex.: $\{e_n\}_{n=1}^{d}$, with $e_n = (0, 0, \dots, 0, 1, 0, \dots, 0)$ ← n^{th} index

is an ONB in $\mathbb{R}^d$ (or $\mathbb{C}^d$).

5.9: Existance of an ONB

Recall: H seperable if it contains a countable dense subset.

Theorem: Gram-Schmidt

$x_1, \dots, x_d$ lin. indep. vectors in a Hilbert space H. Then $\exists$ $e_1, \dots, e_d$ orthon. vectors s.t.: $\text{span}\{e_1, \dots, e_k\} = \text{span}\{x_1, \dots, x_k\}$ $k = 1, \dots, d$.

proof:

1) Let $e_1 = \frac{x_1}{\|x_1\|}$. Then $\text{span}\{e_1\} = \text{span}\{x_1\} = M_1$

2) Now consider $x_2 \notin \text{span}(x_1) = M_1$

We have $x_2 = p_2 + q_2$, $p_2 \in M_1$, $q_2 \in M^\perp$

$\rightarrow$ set $e_2 = \frac{q_2}{\|q_2\|}$

Then $\text{span}\{e_1, e_2\} = \text{span}\{x_1, x_2\}$

⋮

For each $k \le d$

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$\vdots$

For each $k \le d$, at step k, we let $x = p_k + q_k$

p_k: orthon. proj. on $M_{k-1} = \text{span}\{x_1, \dots, x_{k-1}\}$
$= \text{span}\{e_1, \dots, e_{k-1}\}$

Take $e_k = \frac{q_k}{\|q_k\|}$, then $e_k \perp e_j$ for for $j-1, k-1$,
and $\text{span}\{e_1, \dots, e_k\} = \text{span}\{x_1, \dots, x_k\}$

Notice: $p_k = \sum_{j=1}^{k-1} \langle x, e_j \rangle e_j$

Summarized: $\{x_1, \dots, x_d\}$ linearly independent

i) set $e_1 = \frac{x_1}{\|x_1\|}$

ii) At k^{th} step: set $q_k = x_k = \sum_{j=1}^{k-1} \langle x, e_j \rangle e_j \rightarrow e_k = \frac{q_k}{\|q_k\|}$

Example:

Consider $\{t^n\}_{n=0}^{d} = \{1, t, t^2, \dots, t^d\}$ on $\left(\begin{matrix} C[-1,1] \\ L^2[-1,1] \end{matrix}, \|\cdot\|_2 \right)$

Lin. indep., but not orthon.

Gram-Schmidt:

$$e_0(t) = \frac{1}{\|1\|_2} = \frac{1}{\sqrt{2}}, \quad \|1\|_2 = \left(\int_{-1}^{1} 1^2 dt \right)^{\frac{1}{2}}$$

$$p_1(t) = \langle t, e_0 \rangle e_0(t) = \underbrace{\int_{-1}^{1} t \frac{1}{\sqrt{2}} dt}_{\text{odd} \to =0} \cdot e_0(t) = 0$$

$$e_1(t) = \frac{q_1}{\|q_1\|_2} = \frac{t}{\|t\|_2}, \quad \|t\|_2 = \left(\int_{-1}^{1} t^2 dt \right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$$

$$e_1(t) = \sqrt{\frac{3}{2}}\, t$$

$$p_2(t) = \underbrace{\langle t^2, e_0 \rangle}_{\neq 0} e_0(t) + \underbrace{\langle t^2, e_1 \rangle}_{=0} e_1(t)$$

$$\langle t^2, e_0 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2 dt = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{\sqrt{2}}{3}$$

$$q_2(t) = t^2 - p_2 = t^2 - \frac{1}{3}$$

$$\|q_2\|_2 = \left(\int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt \right)^{\frac{1}{2}} = \dots = \left(\frac{8}{45} \right)^{\frac{1}{2}}$$

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$$e_2(t) = \frac{q_2}{\|q_2\|_2} = \frac{1}{(8/45)^{1/2}}\left(t^2 - \frac{1}{3}\right) = \frac{5}{2\sqrt{2}}(3t^2 - 1)$$

⋮

seq. of orthon. polynomials $\{e_0, \dots, e_d\}$ in $C[-1, 1]$ with:
$\text{span}\{e_0, \dots, e_k\} = \text{span}\{1, t, \dots, t^k\} = P_k$

Corollary:

If H is a finite dimensional Hilbert space, then H contains an ONB $\{e_1, \dots, e_d\}$, where $d = \dim(H)$.

Theorem:

If H is an inf. dim., seperable Hilbert space, then H contains an ONB $\{e_n\}_{n\in\mathbb{N}}$.

proof:

H seperable: contains a countable dense seq. $\{z_k\}_{k\in\mathbb{N}}$

k_1: first index where $z_{k_1} \neq 0$

set $z_{k_1} = x_1$

k_2: first index $> k_1$ s.t. $z_{k_2} \notin \text{span}\{x_1\}$

set $z_{k_2} = x_2$

k_3: first index $> k_2$ s.t. $z_{k_3} \notin \text{span}\{x_1, x_2\}$

⋮

and so on...

This gives a lin indep seq. $\{x_n\}_{n\in\mathbb{N}}$ with $\text{span}\{x_n\}_{n=1}^{\infty} = \text{span}\{z_n\}_{n=1}^{\infty}$

Now apply Gram-Schmidt to $\{x_n\}$ to detain an orthon. seq. $\{e_n\}$ with $\text{span}\{e_1, \dots, e_k\} = \text{span}\{x_1, \dots, x_k\} = \text{span}\{z_1, \dots, z_{n_k}\}$ $\forall\ k = 1, 2, \dots$

Recall: $\{z_k\}_{k\in\mathbb{N}}$ is dense $\Rightarrow$ $\{e_n\}_{n\in\mathbb{N}}$ dense

$\Rightarrow$ $\{e_n\}$ is an ONB in H (bc. orthon. and complete). ∎

5.11: The complex trigonometric system

$(C[0,1], \langle\cdot,\cdot\rangle)$ with $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$ is not complete, but $C[0,1] \subseteq L^2[0,1]$, and $(L^2[0,1], \langle\cdot,\cdot\rangle)$ is complete

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$f \in C[0,1]$: $f : [0,1] \longrightarrow \mathbb{C}$ is cont.

Consider the exponential system:

$$\mathcal{E} = \{e_n\}_{n \in \mathbb{Z}} = \{e^{2\pi i n t} : n \in \mathbb{Z}\}$$

Note: i) $|e_n(t)| = |e^{2\pi i n t}| = 1$

$$= |\cos(2\pi n t) + i \sin(2\pi n t)|$$
$$= (\cos^2(2\pi n t) + \sin^2(2\pi n t))^{\frac{1}{2}}$$

ii) e_n is 1-periodic:

$$e_n(t+1) = e^{2\pi i n(t+1)} = \underbrace{e^{2\pi i n t}}_{= e_n(t)} \underbrace{e^{2\pi n i}}_{=1} = e_n(t)$$

iii) $e_n \perp e_k$ for $n \neq k$

$$\langle e_n, e_k \rangle = \int_0^1 e^{2\pi i n t} e^{-2\pi i k t} \, dt = \int_0^1 e^{2\pi i t(n-k)} \, dt$$
$$= \frac{1}{2\pi i (n-k)} e^{2\pi i (n-k) t} \Big|_{t=0}^{t=1} = 0$$

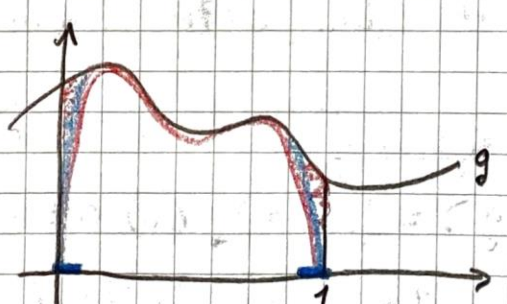
$\rightarrow$ The seq. $\{e_n\}$ is orthon., bc:

$$\|e_n\|_2^2 = \int_0^1 |e^{2\pi i n t}|^2 \, dt = 1$$

<u>Claim:</u>

$C[0,1] \subseteq L^2[0,1]$ is a <u>dense</u> subset. i.e. $\forall \varepsilon > 0$ $\forall f \in L^2[0,1]$ $\exists g \in C[0,1]$ s.t. $\|f - g\|_2 < \varepsilon$.

Step 2: Approx. $g \in C[0,1]$ by periodic h, i.e. $h \in C[0,1]$ where $h(0) = h(1)$ and $\|h - g\|_2 < \varepsilon$.



$$h(t) = \begin{cases} 0 & t = 0 \\ \text{linear} & 0 < t < \delta \\ g(t) & \delta < t < 1-\delta \\ \text{linear} & 1-\delta < t < 1 \\ 0 & t = 1 \end{cases}$$

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step 3: Recall the Stone-Weierstrass thm. for periodic fncs.:

$\exists\ N > 0$ and scalars c_n s.t. for:

$P_N = \sum_{n=N}^{N} c_n e^{2\pi i n t}$ we have: $\|h - P_N\|_2 < \varepsilon$

combined: $\|f - P_N\|_2 = \|f - g + g - h + h - P_N\|_2 \leq \|f-g\|_2 + \|g-h\|_2 + \|h - P_N\|_2 < 3\varepsilon$

$\Rightarrow$ $\mathcal{E}$ is dense in $L^2[0,1]$, so $\mathcal{E} = \{e^{2\pi i n t}\}_{n\in\mathbb{Z}}$ is an ONB in $L^2[0,1]$.

$\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(t) e^{-2\pi i n t}\, dt$

n^{th} Fourier coefficient of f.

Theorem: (Fourier series for $L^2[0,1]$)

For each $f \in L^2[0,1]$, we have

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n,$$

where this series converges in the $\|\cdot\|_2$-norm. Moreover;

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \quad \forall f \quad \text{(Parseval's equality)}$$

and

$$\langle f, g \rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\hat{g}(n) \quad \forall f, g \in L^2[0,1] \quad \text{(Plancherel)}$$

Remark:

Convergence of $\sum \hat{f}(n) e^{2\pi i n t}$ to f is in <u>L^2-norm</u>, i.e. <u>not</u> necessarily pointwise.