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Lemma: $\mathcal{M} \subseteq H$ closed subspace (H: Hilbert space)

Given $x, p \in H$, TFAE:

i) p is the orthogonal projection of x onto $\mathcal{M}$

ii) $p \in \mathcal{M}$ and $x - p \in \mathcal{M}^\perp$

⊛ iii) $x = p + e$, $p \in \mathcal{M}$ and $e \in \mathcal{M}^\perp$ (Projection thm.)

iv) $e = x - p$ is the orthogonal projection of x onto $\mathcal{M}^\perp$.

proof: i => ii)

(case $\mathbb{F} = \mathbb{R}$)

Suppose p is the ortho proj. of x onto $\mathcal{M}$.

Need to show: $e = x - p \in \mathcal{M}^\perp \iff \langle e, y\rangle = 0 \;\forall y \in \mathcal{M}$

Choose any $y \in \mathcal{M}$.

any scalar

$$\Rightarrow p + \lambda y \in \mathcal{M}$$

$$\cancel{\|e\|^2} = \|x - p\|^2 \leq \|x - (p + \lambda y)\|^2 = \|e - \lambda y\|^2 = \langle e - \lambda y, e - \lambda y\rangle$$

$$= \cancel{\|e\|^2} - 2\langle \lambda y, e\rangle + |\lambda|^2 \|y\|^2$$

$$\Rightarrow \forall \lambda \in \mathbb{R}, \quad 2\lambda \langle y, e\rangle \leq |\lambda|^2 \|y\|^2$$

If $\lambda > 0$: $\quad 2\langle y, e\rangle \leq \lambda \|y\|^2 \xrightarrow{\lambda \to 0^+} \langle y, e\rangle \leq 0$

If $\lambda < 0$: $\quad 2\langle y, e\rangle \geq \lambda \|y\|^2 \xrightarrow{\lambda \to 0^-} \langle y, e\rangle \geq 0$

$$\Rightarrow \langle y, e\rangle = 0 \Rightarrow e \in \mathcal{M}^\perp$$

■

Lemma:

a) If $\mathcal{M} \subseteq H$ (Hilbert space) is a closed subspace, then $(\mathcal{M}^\perp)^\perp = \mathcal{M}$

b) If $A \subseteq H$ is <u>any</u> subset, then:

$$A^\perp = \operatorname{span}(A)^\perp = \overline{\operatorname{span}}(A)^\perp \text{ and } (A^\perp)^\perp = \overline{\operatorname{span}}(A), \quad \underline{A \subseteq (A^\perp)^\perp}$$

proof: of a):

$x \in \mathcal{M} \Rightarrow \langle x, y \rangle = 0 \quad \forall \quad y \in \mathcal{M}^\perp$

$\Rightarrow x \in (\mathcal{M}^\perp)^\perp \Rightarrow \mathcal{M} \subseteq (\mathcal{M}^\perp)^\perp$

$x \in (\mathcal{M}^\perp)^\perp$, then $x = p + e$, $x \in \mathcal{M}$, $e \in \mathcal{M}^\perp$.

Have seen: $\mathcal{M} \subseteq (\mathcal{M}^\perp)^\perp \Rightarrow p \in \underbrace{(\mathcal{M}^\perp)^\perp}_{\text{subspace}}$

$\Rightarrow \underbrace{x - p}_{e} \in (\mathcal{M}^\perp)^\perp$

$\Rightarrow e \in \mathcal{M}^\perp$ and $e \in (\mathcal{M}^\perp)^\perp$

$\Rightarrow \langle e, e \rangle = 0 \Rightarrow e = 0$

$\Rightarrow x = p \in \mathcal{M}$

$\Rightarrow (\mathcal{M}^\perp)^\perp \subseteq \mathcal{M} \Rightarrow (\mathcal{M}^\perp)^\perp = \mathcal{M}$ ■

Corollary:

Given a seq. $\{x_n\}$ in a Hilbert space H, we have:

span $\{x_n\}$ is dense in H. $\Leftrightarrow$ The only vector in H ortho. to all x_n is the zero-vector:

$\langle x, x_n \rangle = 0 \ \forall \ n \in \mathbb{N} \Rightarrow x = 0$

proof:

$\mathcal{M} = \overline{\text{span}}\{x_n\} \subseteq H$

$\mathcal{M}^\perp = \{x_n\}^\perp$

Know: $H^\perp = \{0\}$

span $\{x_n\}$ is dense in H

$\Updownarrow$

$\mathcal{M} = H$

$\Updownarrow$

$\mathcal{M}^\perp = H^\perp = \{0\}$

$\Updownarrow$

$\mathcal{M}^\perp = \{x_n\}^\perp = \{0\}$ ■

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5.7: Orthogonal and orthonormal spaces

We will see: Hilbert space H

orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$.

Then $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \quad \forall \ x \in H$.

Theorem:

Let H be a Hilbert space.

Let $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ be an orthonormal seq. and:

$\mathcal{M} = \overline{\text{span}}(\mathcal{E})$

The following holds:

a) Bessel's inequality:

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 \quad \forall \ x \in H$$

b) If $x = \sum_{n=1}^{\infty} c_n e_n$ converges, then $c_n = \langle x, e_n \rangle \ \forall n$.

c) $\sum c_n e_n$ converges $\iff \sum |c_n|^2 < \infty$

d) If $x \in H$, then $p = \sum \langle x, e_n \rangle e_n$ is the ortho. proj. of x onto $\mathcal{M}$ (meaning $x - p \in \mathcal{M}^\perp$) and $\|p\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$

e) For $x \in H$, we have:

$$x \in \mathcal{M} \iff x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \iff \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$$

proof:

a) Choose $x \in H$. Let:

$$P_N = \sum_{n=1}^{N} \langle x, e_n \rangle e_n$$

$$q_N = x - P_N \perp P_N$$

By Pythagoras:

$$\|P_N\|^2 = \sum_{n=1}^{N} \|\langle x, e_n \rangle e_n\|^2 = \sum_{n=1}^{N} |\langle x, e_n \rangle|^2$$

$$q_N \perp p_N: \langle p_N, q_N \rangle = \langle p_N, x - q_N \rangle$$

$$= \langle \sum_1^N \langle x, e_n \rangle e_n, x - \sum_{n=1}^N \langle x, e_n \rangle e_n \rangle$$

$$= \sum_{n=1}^N \langle x, e_n \rangle \overline{\langle x, e_n \rangle} - \|p_N\|^2$$

$$= \sum_{n=1}^N |\langle x, e_n \rangle|^2 - \|p_N\|^2 = 0$$

Thus, for $x = q_N + p_N$, we get:

$$\|x\|^2 = \|p_N + q_N\|^2 \underset{\substack{\uparrow \\ \text{Pythagoras}}}{=} \|p_N\|^2 + \|q_N\|^2 \geq \|p_N\|^2 = \sum_{n=1}^N |\langle x, e_n \rangle|^2$$

$$\overset{N \to \infty}{\Longrightarrow} \|x\|^2 \geq \sum_{n=1}^\infty |\langle x, e_n \rangle|^2$$

b) $x = \sum_{n=1}^\infty c_n e_n$ converges $\Longrightarrow \langle x, e_m \rangle = c_m$:

$$\langle x, e_m \rangle = \langle \sum_{n=1}^\infty c_n e_n, e_m \rangle = \sum_{n=1}^\infty c_n \underbrace{\langle e_n, e_m \rangle} = c_m$$

c) $x = \sum c_n e_n$ converges $\Longleftrightarrow \sum |c_n|^2 < \infty$:

$x = \sum c_n e_n$ conv. $\overset{b)}{\Longrightarrow} c_n = \langle x, e_n \rangle$

$$\overset{a)}{\Longrightarrow} \sum |c_n|^2 = \sum |\langle x, e_n \rangle|^2 \leq \|x\|^2 < \infty$$

Conversely, say:

$$\sum |c_n|^2 < \infty$$

Set:

$$S_N = \sum_{n=1}^N c_n e_n, \quad t_N = \sum_{n=1}^N |c_n|^2, \quad (t_N)_{N \geq 1} \text{ is Cauchy in } \mathbb{R}$$

Say $N > M$, then:

$$\|S_N - S_M\|^2 = \sum_{n=M+1}^N \|c_n e_n\|^2 = \sum_{n=M+1}^N |c_n|^2 = |t_N - t_M| < \varepsilon$$

for suff. large M.

cont.

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cont. $\Rightarrow (S_N)_{N\geq 1}$ is Cauchy in H and H is complete.

$\Rightarrow S_N \longrightarrow S \in H$

d) By a) and c), we know that:

$\underbrace{\sum \langle x, e_n \rangle e_n}_{p}$ converges.

clear that:

$p \in M = \overline{\text{span}}(\mathcal{E})$

Must show: $x - p \in M^\perp$.

Recall: $\mathcal{E}^\perp = \overline{\text{span}}(\mathcal{E})$, so suff. to show that:

$$\langle x - p, e_k \rangle = 0 \quad \forall k.$$

$$\langle x - p, e_k \rangle = \langle x - \sum \langle x, e_n \rangle e_n, e_k \rangle$$

$$= \langle x, e_n \rangle - \langle \sum \langle x, e_n \rangle e_n, e_k \rangle$$

$$= \langle x, e_n \rangle - \sum_n \langle x, e_n \rangle \underbrace{\langle e_n, e_k \rangle}$$

$$= \langle x, e_k \rangle - \langle x, e_k \rangle = 0$$

$$\|p\|^2 = \langle \sum_n \langle x, e_n \rangle e_n, \sum_k \langle x, e_k \rangle e_k \rangle$$

$$= \sum_k \overline{\langle x, e_k \rangle} \sum_n \langle x, e_n \rangle \langle e_n, e_k \rangle$$

$$= \sum_k |\langle x, e_k \rangle|^2$$

e) $x \in M \Rightarrow$ ortho. proj. of x onto M is x,

$x = \sum \langle x, e_n \rangle e_n$

$\|x\|^2 = \sum |\langle x, e_n \rangle|^2$

$\|x\|^2 = \sum |\langle x, e_n\rangle|^2$

For all $x \in H$, we have

$$x = p + (x-p), \quad p \in \mathcal{M}, \quad (x-p) \in \mathcal{M}^\perp$$

$$\|x\|^2 = \|p\|^2 + \|x-p\|^2 = \sum |\langle x, e_n\rangle|^2 + \|x-p\|^2$$

$$\Rightarrow \|x-p\|^2 = 0 \Rightarrow x = p = \sum \langle x, e_n\rangle e_n \in \mathcal{M}$$

■

5.8: Orthonormal bases

Theorem:

If H is a Hilbert space and $\{e_n\}$ is an orthonormal seq. in H, then TFAE:

a) $\{e_n\}$ is complete in H, i.e. $\overline{\operatorname{span}\{e_n\}} = H$.

b) $\{e_n\}$ is a Schauder basis for H:

$$\forall x \in H \ \exists \text{ unique scalars } c_n \text{ s.t. } x = \sum c_n e_n$$

c) If $x \in H$, then:

$$x = \sum \langle x, e_n\rangle e_n$$

(and this series converges in the induced norm on H).

d) Plancherel's ~~in~~equality:

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \quad \forall x \in H$$

e) Parseval's ~~in~~equality:

$$\langle x, y\rangle = \sum_n \langle x, e_n\rangle\langle e_n, y\rangle \quad \forall x, y \in H$$

Remark:

Completely analogous thm. can be stated for finite orthon. seq. $\{e_1, \dots, e_d\}$ in finite-dimentional space.

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<u>Definition:</u> Orthonormal basis

A countably infinite orthonormal seq.: $\{e_n\}$, that is complete in a Hilbert space H is an <u>orthonormal basis (ONB)</u>.