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$L^2[a,b]$ square integrable fncs. f on $[a,b]$
$\cup$
$C[a,b]$ $\quad \int_a^b |f(t)|^2 dt < \infty$

Lebesgue integral

$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt, \quad f,g \in L^2[a,b]$

$(L^2[a,b], \langle \cdot, \cdot \rangle)$ is complete, so is a Hilbert space.

5.4: Orthogonal and orthonormal sets

Definition: Orthogonality

Let $(H, \langle \cdot, \cdot \rangle)$ be i.p. space, I index set.

a) $x, y \in H$ are orthogonal, $x \perp y$, if $\langle x, y \rangle = 0$

b) $\{x_i\}_{i \in I}$ is an ortogonal set if $x_i \perp y_j \ \forall \ i \neq j$ $(\langle x_i, x_j \rangle = 0)$

c) $\{x_i\}_{i \in I}$ is orthonormal if $\langle x_i, x_j \rangle = 0$ for $i \neq j$ and $\|x_i\| = 1$.

i.e. $\langle x_i, x_j \rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Ex.:

i) $\{\delta_n\}_{n \geq 1}$ with $\delta_n = (0, 0, \ldots, 0, \underset{n^{th}\text{ index}}{1}, 0, \ldots)$ is an orthonomal seq. in $(\ell^2, \langle \cdot, \cdot \rangle)$.

$\{\delta_{2n}\}_{n \geq 1}$ is also an orthonormal seq.

ii) In $C[-1, 1]$, let $M = \{t^k\}_{k=0}^{\infty}$.

We have $\langle t^{2k}, t^{2j+1} \rangle = \int_{-1}^1 t^{2k} \overline{t^{2j+1}}\,dt = \underbrace{\int_{-1}^1 t^{2k+2j+1}\,dt}_{\text{odd fnc.}} = 0$,

but M is not an orthogonal seq.

$\langle t^{2k}, t^{2j} \rangle = \int_{-1}^1 t^{2k+2j}\,dt = \frac{2}{2j+2k+1} \neq 0$.

However, the seq. is lin. indep. $\longrightarrow$ Can use Gram-Schmidt to construct an orthogonal seq.

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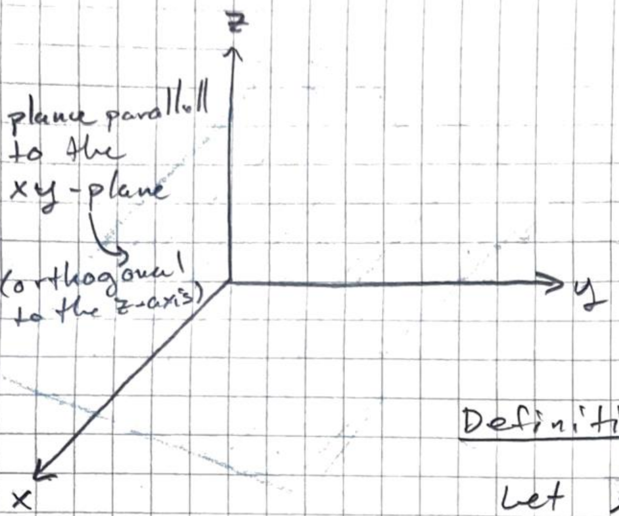
§5. Orthogonal complements



Setting:

Inner product space $(H, \langle \cdot, \cdot \rangle)$.

$A, B \subseteq H$ subsets.

$x \perp A$ means $\langle x, y \rangle = 0 \ \forall y \in A$.

$A \perp B$ means $\langle x, y \rangle = 0 \ \forall x \in A, y \in B$

Definition: Orthogonal complement

Let $A \subseteq H$. The orthogonal complement of A is the set:

$$A^\perp = \{ x \in H : \langle x, y \rangle = 0 \ \forall y \in A \}$$

Ex.:

Consider $(\ell^2, \langle \cdot, \cdot \rangle)$.

odd $\to$ $\mathcal{O} = \{ x \in \ell^2 : x = (x_1, 0, x_3, 0, x_5, \dots) \}$

even $\to$ $E = \{ x \in \ell^2 : x = (0, x_2, 0, x_4, 0, \dots) \}$

If $x \in E$, then $\langle x, y \rangle = 0 \ \forall y \in \mathcal{O}$

$\Rightarrow E \subseteq \mathcal{O}^\perp$

On the other hand, if $x \in \mathcal{O}^\perp$, then $\langle x, y \rangle = 0 \ \forall y \in \mathcal{O}$.

In particular $\langle x, \delta_{2k+1} \rangle = x_{2k+1} = 0 \ \forall k \in \mathbb{N}$

$\Rightarrow x \in E \Rightarrow \mathcal{O}^\perp \subseteq E \Rightarrow \mathcal{O}^\perp = E$

Lemma:

$(H, \langle \cdot, \cdot \rangle)$ inner product space and $A \subseteq H$ subset.

a) $A^\perp$ is a closed <u>subspace</u> of H. $\leftarrow$ Problem set

b) $H^\perp = \{0\}$, $\{0\}^\perp = H$.

c) If $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$

d) $A \subseteq (A^\perp)^\perp$

(and if A is a closed subspace of H, then $A = (A^\perp)^\perp$)

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5.6: orthogonal projections and the closest point theorem

Q: Given subset $S \subseteq X$ and a point $x \notin S$, when can we find $y \in S$ that is „closest to x"?

Definition:

$(X, \|\cdot\|)$ normed space,

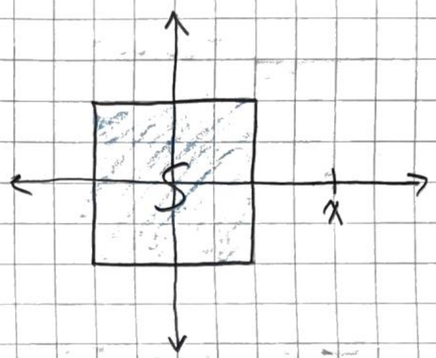
$S \subseteq X$ subset.

The distance from S to $x \in X$ is $\operatorname{dist}(x, S) = \inf_{y \in S} \|x - y\|$

$y \in S$ is a <u>closest point</u> to x (in S) if:

$\|x - y\| \leq \|x - z\| \;\; \forall \; z \in S$.

Ex.:



$S = [-1, 1]^2 \subseteq \mathbb{R}^2$, $x = (2, 0)$

If $\|\cdot\|_\infty$: $\operatorname{dist}(x, s) = 1$ and any

$z_t = (1, t)$ is a closest point.

If $\|\cdot\|_2$: $\operatorname{dist}(x, s) = 1$ and $z = (1, 0)$ is a <u>unique</u> closest point.

<u>Theorem:</u> Closest point theorem

$(H, \langle \cdot, \cdot \rangle)$ Hilbert space,

$S \subseteq H$ nonempty, closed and convex. Given any $x \in H$

$\exists$ <u>unique</u> $y \in S$ that is closest to x, i.e. s.t.:

$\|x - y\| = \operatorname{dist}(x, s) = \inf_{z \in S} \|x - z\|$

<u>proof:</u>

Three „facts":
1) H Hilbert
2) S closed
3) S convex ←

Set $d = \operatorname{dist}(x, S)$

Take $(y_n) \subseteq S$ s.t.:

$\|x - y_n\| \to d$ as $n \to \infty$.

Moreover, $\|x - y_n\| \geq d \;\; \forall \; n$.

Thus, for fixed $\varepsilon > 0$, $\exists \; N > 0$ s.t. $d^2 \leq \|x - y_n\|^2 < d^2 + \varepsilon^2$

$\forall \; n > N$.

cont.

cont. → Now, let $p = \frac{1}{2}(y_n + y_m)$

S convex, ($tx - (1-t)y \in S \ \forall x, y \in S$)

so $p \in S \Rightarrow \|x - p\| \geq \text{dist}(x, S) = d$.

Hence, for $n, m \geq N$, we have:

$$\|y_n - y_m\|^2 + 4d^2 \leq \|y_n - y_m\|^2 + 4\|x - p\|^2$$

$$= \|y_n - y_m\|^2 + \cancel{4}\left\|2x - \frac{y_n - y_m}{\cancel{2}}\right\|^2$$

$$= \|(x - y_m) - (x - y_n)\|^2 + \|(x - y_m) + (x - y_n)\|^2$$

▱-law → $= 2(\|x - y_n\|^2 + \|x - y_m\|^2) \leq 2(2(d^2 + \varepsilon^2))$

$$= 4(d^2 + \varepsilon^2)$$

$\Rightarrow \|y_n - y_m\| \leq 2\varepsilon \Rightarrow (y_n)$ is Cauchy in H.

H is a Hilbert space, so $y_n \to y \in H$.

S is closed and $(y_n) \subseteq S \to y \in S$.

Finally, by the continuity of the norm, since $x - y_n \longrightarrow x - y$, we get:

$$\|x - y\| = \lim_{n \to \infty} \|x - y_n\| = d$$

$\Rightarrow y$ is a point in S closest to x.

<u>Uniqueness of y:</u>

Assume $\|x - y\| = \|x - z\| = d$ for $z \in S$.

Midpoint $p = \frac{y + z}{2} \in S \Rightarrow \|x - p\| \geq d$

$$\cancel{2 \cdot 2}d^2 = 2(\|x - y\|^2 + \|x - z\|^2)$$

▱-law → $= \|(x - y) - (x - z)\|^2 + \|(x - y) + (x - z)\|^2$

$$= \|y - z\|^2 + \|x - p\|^2 \geq \|y - z\|^2 + \cancel{4}d^2$$

cont.

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cont.

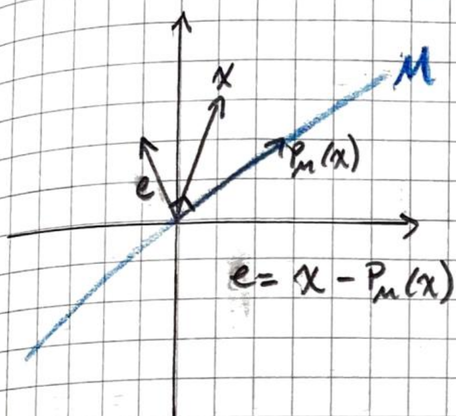
$$\Longrightarrow \|y-z\|^2 = 0 \iff y = z \quad \blacksquare$$

Definition: Orthogonal projection

Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$.

a) Given $x \in \mathcal{H}$, the unique $p \in \mathcal{M}$ closest to x is called the <u>orthogonal projection</u> of x onto $\mathcal{M}$.

b) The func. $P_{\mathcal{M}} : \mathcal{H} \to \mathcal{H}$ defined by $P_{\mathcal{M}}(x) = p$ is called the <u>orthogonal projection</u> of $\mathcal{H}$ on $\mathcal{M}$.



$e = x - P_{\mathcal{M}}(x)$

<u>Lemma:</u>

$\mathcal{M} \subseteq \mathcal{H}$ closed subspace

$\uparrow$ Hilbert space.

T.F.A.E. for $x, p \in \mathcal{H}$:

a) p is the ortho. proj. of x onto $\mathcal{M}$

$(P_{\mathcal{M}}(x) = p)$.

b) $p \in \mathcal{M}$ and $x - p \in \mathcal{M}^{\perp}$

* c) <u>Projection theorem:</u>

$x = p + e, \; p \in \mathcal{M}, \; e \in \mathcal{M}^{\perp}$

d) e is the ortho. proj. of x onto $\mathcal{M}^{\perp}$.

Tuesday:

proof: a) $\Rightarrow$ b)