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Exs.: $\mathbb{R}^d$ $\quad \|\cdot\|_\infty, \|\cdot\|_1 \leftarrow$ equiv

$\ell^1 \quad \|\cdot\|_\infty, \|\cdot\|_1 \leftarrow$ not equiv.

Theorem: If X is a finite-dimensional vector space (e.g. $\mathbb{R}^d$ or $\mathbb{C}^d$), then all norms are equivalent.

Ch. 4: Further results on Banach spaces

(Selected topics from ch. 4.1 - 4.6)

4.1: Infinite series in Normal spaces

$x_1, x_2, \ldots \in X, \quad \sum_{n=1}^{\infty} x_n = ?$

$S_N = \sum_{n=1}^{N} x_n \longrightarrow ?$ as $N \to \infty$

$\exists\, x \in X$ s.t. $\|S_N - x\| \to 0$ as $N \to \infty$?

Definition: (convergent series)

Let $(x_n) \subset X$. We say that the infinite series $\sum x_n$ converges if $\exists\, x \in X$ s.t.:

$S_N = \sum_{n=1}^{N} x_n$ converges to x as $N \to \infty$.

i.e. meaning:

$\lim_{N\to\infty} \|S_N - x\| = 0$

Ex.:

i) Let $\delta_n = (0, 0, \ldots, 0, \underset{\uparrow\, n\text{th index}}{1}, 0, \ldots) \in \ell^\infty$

Does $\sum_{n=1}^{\infty} \delta_n$ converge?

No! $S_N = \sum_{n=1}^{N} \delta_n = (1, 1, \ldots, 1, 0, 0, \ldots) \to x = (1, 1, \ldots)$

$\|x - S_N\|_\infty = \|(0, 0, \ldots, 0, 1, 1, \ldots)\|_\infty = 1 \not\to 0$

ii) Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \delta_n$ "=" $(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots) = \left(\frac{(-1)^n}{n}\delta_n\right)_{n\in\mathbb{N}}$

In ℓ^1: $x = \left(\frac{(-1)^n}{n}\delta_n\right)_{n\in\mathbb{N}} \notin \ell^1$ because the S_N's are not bounded.

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cont.

$$\|S_N\|_1 = \sum_{n=1}^{N} \left| \frac{(-1)^n}{n} \right| \to \infty \quad \text{as } N \to \infty$$

$$\Rightarrow \sum \frac{(-1)^n}{n} S_n \text{ is not convergent in } \ell^1.$$

In ℓ^p, $1 < p < \infty$: $S_N \to x = \left(\frac{(-1)^n}{n} \delta_n \right)_{n \in \mathbb{N}}$

$$\|S_N\|_p = \sum_{n=1}^{N} \left| \frac{(-1)^n}{n} \right|^p < \infty$$

In general:

1) $\sum c_n \delta_n$ conv. in ℓ^p, $1 < p < \infty$, iff $c = (c_n) \in \ell^p$.

2) $\sum c_n \delta_n$ conv. in ℓ^∞ iff $c = (c_n) \in c_0$

Lemma: (4.2.1):

If $\sum x_n$ is a convergent series in a normed space $(X, \|\cdot\|)$, then:

$$\underbrace{\left\| \sum_{n=1}^{\infty} x_n \right\|}_{x} \leq \sum_{n=1}^{\infty} \|x_n\|$$

proof: $\|x\| = \|\lim_{N \to \infty} S_N\| = \|\lim_{N \to \infty} \sum_{n=1}^{N} x_n\|$

$= \lim_{N \to \infty} \|\sum_{n=1}^{N} x_n\|$ (cont. of norm)

$\leq \lim_{N \to \infty} \sum_{n=1}^{N} \|x_n\|$ (Δ-ineq.)

$= \sum_{n=1}^{\infty} \|x_n\|$ (def. of inf. series)

4.4: Span, closed span and complete sequences

Recall: span $A \subseteq X \leftarrow$ interior space

$$\text{span}(A) = \left\{ \sum_{n=1}^{N} c_n x_n : N > 0, x_n \in A, c_n \in \mathbb{F} \right\}$$

Ex:

i) $\mathcal{M} = \{t^n\}_{t=1}^{\infty}$ monomials

span$(\mathcal{M}) = \mathcal{P}$: space of polynomials

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ii) $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$

$$\text{span}(\mathcal{E}) = \{\sum_{n=1}^{N} c_n \delta_n : N > 0, c_n \in \mathbb{F}\} = c_{00}$$

__Definition:__ (closed span)

$A \subseteq X$

The closed span of A is the closure of span(A):

$$\overline{\text{span}}(A) = \overline{\text{span}(A)} = \{y \in X : \exists (y_n) \subseteq \text{span}(A) \text{ s.t. } y_n \to y\}$$

Eg. any conv. series $\sum c_n x_n$ with $c_n \in \mathbb{F}$, $x_n \in A$ belongs to $\overline{\text{span}}(A)$.

__Definition:__ (complete sequence)

Let (x_n) be a seq in $(X, \|\cdot\|)$, then (x_n) is a __complete seq__ if span(x_n) is dense.

__Remark:__

i) Not to be confused with complete space.

ii) Also known as __total-__ or __fundamental__ sequence.

4.5: Hamel vs. Schauder Bases

__Recall:__ 1) $A \subseteq X$ lin indep. set of vectors if:

$$\sum_{n=1}^{N} c_n x_n = 0 \iff c_1 = \dots = c_n = 0$$

$(c_1, \dots, c_N \in \mathbb{F},\ x_1, \dots, x_n \in A)$

2) Hamel basis: A set $\mathcal{B} \subseteq X$ s.t.:

i) $\mathcal{B}$ is lin. indep.

ii) span$(\mathcal{B}) = X$

3) Finite Hamel basis $\mathcal{B}$

↳ X finite dimensional and $\dim X = \#\mathcal{B}$

__Theorem:__ (4.5.3)

If $(X, \|\cdot\|)$ is an inf-dim Banach space and $\mathcal{B}$ is a Hamel basis for X, then $\mathcal{B}$ is __uncountable__. ↗

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Definition: (Schauder basis)

Let $(X, \|\cdot\|)$ be a Banach space. A countably infinite seq. (x_n) of elements of X is a Schauder basis for X if for each $x \in X$ there exists unique scalars $c_n(x)$ s.t.:

$$x = \sum_{n=1}^{\infty} c_n(x) x_n$$

where this series is convergent in $\|\cdot\|$.

Ex:

Consider $\mathcal{E} = \{\delta_n\}_{n \in \mathbb{N}}$ (std. unit vectors) in ℓ^1

i) $\mathcal{E}$ is a Schauder basis for ℓ^1

For $x = (x_1, x_2, \dots) \in \ell^1$, one can show that

$$x = \sum_{n=1}^{\infty} x_n \delta_n \qquad S_N = \sum_{n=1}^{N} x_n \delta_n$$

is the unique series representation of x

$(\|S_N - x\|_1 \to 0)$

ii) $\mathcal{E}$ is not a Hamel basis for ℓ^1.

$$\text{span}(\mathcal{E}) = \left\{ \sum_{n=1}^{N} c_n \delta_n : N > 0,\ c_i \in \mathbb{F} \right\}$$

$$= \left\{ (x_1, \dots, x_N, 0, \dots, 0) : N > 0,\ c_i \in \mathbb{F} \right\}$$

$$= c_{00} \subsetneq \ell^1$$

■

4.6: Weierstrass Approximation

Theorem: Fnc f which is k times differentiable at pt. a

$$\Rightarrow f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

for x close to a.

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Suppose $f \in C[a,b]$, can we approx. f by a polynomial p s.t.:

$$\|f - p\|_u = \sup_{a \le t \le b} |f(t) - p(t)|$$

is arbitrarily small?

Theorem: (Stone-Weierstrass thm.)

If $f \in C[a,b]$ and $\varepsilon > 0$, then $\exists$ polynomial

$$p(t) = \sum_{n=0}^{N} c_n t^n \quad \text{s.t.} \quad \|f - p\|_u = \sup_{a \le t \le b} |f(t) - p(t)| < \varepsilon$$

Remark:

Suppose $f \in C(\mathbb{R})$ is 1-periodic. Then for any $\varepsilon > 0$ $\exists$ trigonometric polynomial

$$t_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi n x} \quad \text{s.t.} \quad |f - t_N| < \varepsilon$$

5: Inner Products and Hilbert Spaces

„Inspiration": $\mathbb{R}^d$, x, y
- $\|x\|_2 = (|x_1|^2 + \cdots + |x_d|^2)^{\frac{1}{2}}$ (length)
- $\|x - y\|_2$ (distance)
- „angle" between x, y. In particular, they are perpendicular if $x \cdot y = 0$.

5.1: Definition of inner product

$x, y \in \mathbb{F}^d$

Dot product: $x \cdot y = x_1 \overline{y_1} + \cdots + x_d \overline{y_d}$

Properties: $x \cdot x = |x_1|^2 + \cdots + |x_d|^2 \ge 0$

$x \cdot x = 0 \iff x = 0$

$x \cdot y = \overline{y \cdot x}$

$(ax + by) \cdot z = a(x z) + b(y z)$

Def.: H vector space. An inner product on H is a func $\langle \cdot, \cdot \rangle$, $H \times H \to \mathbb{F}$ s.t. $\forall\, x, y, z \in H$ $\forall\, a, b \in \mathbb{F}$, we have:

a) $\langle x, x \rangle \ge 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

b) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

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c) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

We say that $(H, \langle \cdot, \cdot \rangle)$ is an inner product space.

Remark:

If $\langle \cdot, \cdot \rangle$ is an inner product on H, then:

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}, \qquad x \in H$$

defines a norm on H. This is the norm induced by $\langle \cdot, \cdot \rangle$

Exercise: check this!

Ex.:

Usual dot product is an inner product on $\mathbb{R}^d$:

$$x \cdot y = \sum_{i=1}^{d} x_i \cdot y_i, \qquad x, y \in \mathbb{R}^d$$

Dot product is also an inner product on $\mathbb{C}^d$:

$$x \cdot y = \sum_{i=1}^{d} x_i \overline{y_i}, \qquad x, y \in \mathbb{C}^d$$