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Recall: $f: X \longrightarrow Y$

Injective: $f(x_1) = f(x_2) \implies x_1 = x_2$

Surjective: For all $y \in Y$ there is $x \in X$ s.t. $f(x) = y$.

Bijective: Injective and surjective

- If $f: X \to Y$ is a bijection, then there is an inverse: $f^{-1}: Y \to X$ given by $f^{-1}(y) = x$ if $f(x) = y$.

- Identity function/map: $id_X : X \longrightarrow X$

$$id_X(x) = x \text{ for all } x \in X$$

- The <u>composition</u> of a function $g: Y \to Z$ with $f: X \to Y$ is the function $g \circ f: X \to Z$ defined by: $(g \circ f)(x) = g(f(x))$ for $x \in X$.

<u>Def.:</u> (Alternative version)

Given a bijection $f: X \to Y$, the inverse function: $f^{-1}: Y \to X$ is the <u>unique</u> func. satisfying: $f^{-1} \circ f = id_X$ <u>and</u> $f \circ f^{-1} = id_Y$.

(Exercise: show that such a func. <u>must</u> be unique).

- If $f: X \to \mathbb{R}$, then f is <u>real-valued</u>.

- If $f: X \to \mathbb{C}$, then f is <u>complex-valued</u>.

Example: $\frac{d}{dx} : C^1(\mathbb{R}, \mathbb{R}) \longrightarrow C(\mathbb{R}, \mathbb{R})$

Injective? No; Let $g \in C^1(\mathbb{R}, \mathbb{R})$. Then $\frac{d}{dx}(g + c) = \frac{d}{dx}(g)$

where c is a constant.

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Surjective? Yes; given any $f \in C(\mathbb{R}, \mathbb{R})$, we can define: $F \in C^1(\mathbb{R}, \mathbb{R})$ by $F: x \to \int_0^x f(t)dt$ and $\frac{d}{dx} F = f$ (Fundamental theorem of calculus)

Cardinality (1.4):

Cardinality: Tool for comparing the sizes of sets.

Def.: We say that two sets A and B have the <u>same cardinality</u> if there exists a bijection between A and B

Ex.:

i) The intervals $[0,2]$ and $[0,1]$ have the same cardinality

$$f: [0,2] \to [0,1] \quad , \quad f(t) = \frac{t}{2}$$

ii) $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N} \setminus \{1\} = \{2, 3, 4, \dots\}$ have the same cardinality; $f(n) = n+1$.

iii) n finite integer, then there is no bijection $f: \{1, 2, \dots, n\} \to \mathbb{N}$.
i.e.: these sets <u>do not</u> have the same cardinality.

Def.: Let X be a set. We say X is <u>finite</u> if either $X = \emptyset$ or there exists $n \in \mathbb{N}$ s.t. X has the same cardinality $\{1, 2, \dots, n\}$, i.e. $\exists$ bijection $f: \{1, 2, \dots, n\} \to X$ for some n.
X is <u>infinite</u> if it is not finite

Def.: A set X is:

i) countably infinite if it has the same cardinality as $\mathbb{N}$, $\exists$ bijection $f: X \to \mathbb{N}$ (or opposite).

ii) countable if it is either countably infinite or finite

(Equivalent: if $\exists$ injection $f: X \to \mathbb{N}$ or $\exists$ surjection $f: \mathbb{N} \to X$)

iii) uncountable if it is not countable

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Ex.: (Countable sets)

i) Any finite set, eg. $\{2, 5, 9\}$

ii) $X = \{1, 4, 9, 16, \dots, n^2, \dots\}$, $f: \mathbb{N} \to X$, $f(n) = n^2$

iii) $\mathbb{N} \times \mathbb{N}$ is countable; we arrange $\mathbb{N} \times \mathbb{N}$ in a table:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \dots \\ (2,1) & (2,2) & (2,3) & \dots & \\ (3,1) & (3,2) & (3,3) & \dots & \\ (4,1) & (4,2) & (4,3) & \dots & \end{matrix}$$

$f(1) = (1,1)$
$f(2) = (2,1)$
$f(3) = (1,2)$
$f(4) = (3,1)$

proof in book iv) $\mathbb{Z}$ and $\mathbb{Q}$ are countable (problem set 1)

v) If X and Y are countable sets, then so is $X \cup Y$ (proof: exersice).

Schröder-Bernstein theorem: (NB: Not in book)

Let X and Y be two sets. Suppose there are injective maps $f: X \to Y$ and $g: Y \to X$. Then there exists a bijection between X and Y.

$(0,1)$ $\boxed{[0,1]}$

Ex: The interval $(0,1) \subseteq \mathbb{R}$ is uncountable.

claim: is uncountable

proof: (the Cantor diagonalization argument) suppose that $(0,1)$ is countable.

$$(0,1) = \{\underset{f(1)}{x_1}, \underset{f(2)}{x_2}, \underset{f(3)}{x_3}, x_4, \dots\}$$

$f: \mathbb{N} \to (0,1)$

$x_i = 0, x_{i1}, x_{i2}, x_{i3}, \dots$

Now let $a = 0, a_1 a_2 a_3 a_4 a_5 \dots$ where $a_i = \begin{cases} 3 & \text{if } x_{ii} \neq 3 \\ 1 & \text{if } x_{ii} = 3 \end{cases}$

Then $a_i \neq x_{ii}$, so by construction $a\, x = x_i$ for all i.

Moreover, we must have $a \in (0,1)$.

This is a contradiction, so $(0,1)$ cannot be countable. ■

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<u>Ex.:</u> The set of all binary sequences

$$X = \{(x_1, x_2, x_3, \dots) : x_i \in \{0,1\}\}$$

is countable.

proof: Problem set 2.

■

Lemma: Let X and Y be sets.

i) If X is countable $Y \subseteq X$, then Y is also countable.

$$\{1, 2, 3, 4, 5, \dots\} \longrightarrow \{x_1, x_2, x_3, x_4, \dots\}$$

ii) If X is uncountable and $X \subseteq Y$, then Y is uncountable.

iii) If X is countable and there is an injection

$$f : Y \to X,$$

then Y is countable.

iv) If X is uncountable and $\exists$ injective

$$f : X \to Y,$$

then Y is uncountable.

<u>Ex.:</u> Have proved formally that:

$(0,1)$ is uncountable $\overset{ii}{\Longrightarrow}$ $\mathbb{R}$ must be uncountable

$\mathbb{R} \subset \mathbb{C}$ $\overset{ii}{\Longrightarrow}$ $\mathbb{C}$ is uncountable.

<u>Ex.:</u> $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$

Know: $\mathbb{Q}$ is countable

Assume: $\mathbb{I}$ is countable

$\Longrightarrow \mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is countable.

contradiction; so $\mathbb{I}$ is uncountable.

■