BANACH'S FIXED POINT THEOREM AND APPLICATIONS

Banach's Fixed Point Theorem, also known as *The Contraction Theorem*, concerns certain mappings (so-called *contractions*) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a *fixed point*, which we will see is a point that is mapped to itself. The theorem also gives an iterative process by which we can obtain approximations to the fixed point along with error bounds.

Definition 1. A **fixed point** of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is

$$Tx = x.$$

Example 1.

i) A translation $x \to x + a$ in $\mathbb{R}$ has no fixed points.
ii) A rotation of the plane has a single fixed point, namely the center of rotation.
iii) The mapping $x \to x^2$ on $\mathbb{R}$ has two fixed points; 0 and 1.
iv) The projection $(x_1, x_2) \to (x_1, 0)$ on $\mathbb{R}^2$ has infinitely many fixed points; all points of the form $(x, 0)$.

Banach's Fixed Point Theorem is an existence and uniqueness theorem for fixed points of certain mappings. As we will see from the proof, it also provides us with a constructive procedure for getting better and better approximations of the fixed point. This procedure is called *iteration*; we start by choosing an arbitrary x_0 in a given set, and calculate recursively a sequence $x_1, x_2, x_3, \ldots$ by letting

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2 \ldots$$

Such iteration procedures appear in nearly every branch of applied mathematics, and Banach's Fixed Point Theorem is often what guarantees convergence of the scheme and uniqueness of the solution.

Definition 2. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a **contraction** on X if there exists a positive constant $K < 1$ such that

$$d\,(T(x), T(y)) \leq K d(x, y) \quad \text{for all } x, y \in X. \tag{1}$$

Geometrically, this means that the images $T(x)$ and $T(y)$ are closer together than the points x and y.

Theorem 2 (**Banach's Fixed Point Theorem**). Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$ (such that $T(x) = x$).

Proof. Let us choose any $x_0 \in X$, and define the sequence (x_n), where

$$x_{n+1} = T(x_n), \quad n = 1, 2, \ldots \tag{2}$$

Our proof strategy will be to show that 1) this sequence is Cauchy; 2) its limit is a fixed point of X; and 3) the fixed point is unique.

Step 1: By (1) and (2), we have that

$$\begin{aligned} d(x_{m+1}, x_m) &= d(T(x_m), T(x_{m-1})) \\ &\leq K d(x_m, x_{m-1}) \\ &= K d(T(x_{m-1}), T(x_{m-2})) \\ &\leq K^2 d(x_{m-1}, x_{m-2}) \\ &\;\vdots \\ &\leq K^m d(x_1, x_0). \end{aligned}$$

Hence by the triangle inequality we get (for $n \geq m$) that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq (K^m + K^{m+1} + \cdots + K^{n-1}) d(x_1, x_0) = K^m \frac{1 - K^{n-m}}{1 - K} d(x_0, x_1), \end{aligned}$$

where in the last equality we have used the summation formula for a geometric series. Since $0 < K < 1$, we have $1 - K^{n-m} < 1$, and consequently

$$d(x_m, x_n) \leq \frac{K^m}{1 - K} d(x_1, x_0). \tag{3}$$

Since $0 < K < 1$ and $d(x_0, x_1)$ are fixed, it is clear that we can make $d(x_m, x_n)$ as small as we please by choosing m sufficiently large (and $n > m$). This proves that (x_n) is Cauchy. Finally, since (X, d) is complete, there exists an $x \in X$ such that $x_n \to x$.

Step 2: To show that x is a fixed point, we consider the distance $d(x, T(x))$. From the triangle inequality and (1), we get

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_m) + d(x_m, T(x)) \\ &= d(x, x_m) + d(T(x_{m-1}), T(x)) \\ &\leq d(x, x_m) + K d(x_{m-1}, x), \end{aligned}$$

and since $x_n \to x$ it is clear that we can make this distance as small as we please by choosing m sufficiently large. We conclude that

$$d(x, T(x)) = 0 \quad \Rightarrow \quad T(x) = x,$$

so $x \in X$ is a fixed point of T.

Step 3: Suppose there are two fixed points $x = T(x)$ and $\tilde{x} = T(\tilde{x})$. Then from (1) it follows that

$$d(x, \tilde{x}) = d(T(x), T(\tilde{x})) \leq K d(x, \tilde{x}),$$

which implies $d(x, \tilde{x}) = 0$ since $0 < K < 1$. Hence $x = \tilde{x}$, and the fixed point x of T is unique. □

Note that for Banach's Fixed Point Theorem to hold, it is crucial that T is a contraction; it is not sufficient that (1) holds for $K = 1$, i.e. that

$$d(T(x), T(y)) \leq d(x, y) \quad \text{for all } x, y \in X.$$

To see this, observe that the maps $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ given by $T_1(x) = x + 1$ and $T_2(x) = x$ both satisfy (1) with $K = 1$. The map T_1 has no fixed points, whereas T_2 has infinitely many.

Corollary 3 (Iterations and error bounds)**.** The iterative sequence (2) with arbitrary $x_0 \in X$ converges (under the assumptions in Banach's Fixed Point Theorem) to the unique fixed point x of T. Error estimates are the **prior estimate**

$$d(x_m, x) \leq \frac{K^m}{1-K} d(x_0, x_1), \tag{4}$$

and the **posterior estimate**

$$d(x_m, x) \leq \frac{K}{1-K} d(x_{m-1}, x_m). \tag{5}$$

The prior error bound (4) can be used at the beginning of a calculation for estimating the number of steps necessary for obtaining a given accuracy. The posterior bound (5) can be used at intermediate stages to check whether we are possibly converging faster than suggested by (4). We see that if two successive iterations x_m and $x_{m+1} = T(x_m)$ are nearly equal, then this guarantees that we are very close to the true fixed point x.

Proof of Corollary 3. The first statement is obvious from the proof of Banach's Fixed Point Theorem. The prior bound (4) follows from (3) by letting $n \to \infty$. Finally let us establish (5). Since x is a fixed point and T is a contraction, we have

$$\begin{aligned} d(x_m, x) &= d(T(x_{m-1}), T(x)) \\ &\leq K d(x_{m-1}, x) \\ &\leq K\left(d(x_{m-1}, x_m) + d(x_m, x)\right), \end{aligned}$$

where in the last step we have used the triangle inequality. Rearranging terms, we arrive at (5). □

A classical application of Banach's Fixed Point Theorem is Newton's method for finding roots of equations. Starting with a differentiable function f and an initial guess x_0 for a root of f, Newton's method suggests

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{6}$$

as a sequence of successively better approximations to the true root of f. We look at a specific example.

Example 4. Consider the equation $f(x) = x^2 - 3$, which we know has two roots, and let us apply Banach's Fixed Point Theorem to determine when we can expect the scheme (6) to converge to $x = \sqrt{3}$. Setting

$$T(x) := x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3}{2x} = \frac{1}{2}(x + \frac{3}{x}),$$

we see that T is a map from the closed set $[\sqrt{3},\infty)$ into itself. Moreover, a point $x \in [\sqrt{3},\infty)$ is a fixed point of T if and only if $f(x)=0$. Finally, we observe that

$$\begin{aligned} d(T(x),T(y)) &= |T(x)-T(y)| \\ &= \frac{1}{2}\left|\left(x+\frac{3}{x}\right)-\left(y+\frac{3}{y}\right)\right| \\ &= \frac{1}{2}|x-y|\cdot\left|1-\frac{3}{xy}\right| \\ &\le \frac{1}{2}|x-y| = \frac{1}{2}d(x,y), \end{aligned}$$

for all $x,y \in [\sqrt{3},\infty)$. Hence, T is a contraction on the complete space $\big([\sqrt{3},\infty),|\cdot|\big)$, and by Banach's Fixed Point Theorem we conclude that the scheme (6) converges to the root $x=\sqrt{3}$ for any starting point $x_0 \in [\sqrt{3},\infty)$.

In fact, the scheme will converge to $x=\sqrt{3}$ for any starting point $x_0 \in (0,\infty)$; one can check that for any $x_0 \in (0,\sqrt{3})$, we have

$$x_1 = T(x_0) = \frac{1}{2}(x+\frac{3}{x}) > \sqrt{3},$$

and we may therefore use Banach's Fixed Point Theorem with the "new" starting point x_1.

1. Applications

The most interesting applications of Banach's Fixed Point Theorem arise in connection with function spaces. The theorem then yields existence and uniqueness results for differential and integral equations, as we will now see.

1.1. **Application to integral equations.**

In this section we consider integral equations of the form

$$f(x) = \lambda \int_a^b k(x,y)f(y)\,dy + g(x), \tag{7}$$

where $f:[a,b]\to\mathbb{R}$ is an unknown function, $k:[a,b]\times[a,b]\to\mathbb{R}$ is a given function (called the **kernel**) and λ is a parameter. Such integral equations can be considered in various function spaces. In this section we consider (7) only in $(C[a,b],d_\infty)$. We assume that $g\in C[a,b]$, and that the kernel k is continuous on the square $[a,b]\times[a,b]$.

Theorem 5. The metric space of continuous functions $C[a,b]$ with the uniform metric d_∞ is complete.

Recall that the uniform metric d_∞ is given by

$$d_\infty(f,g) = \|f-g\|_\infty = \sup_{1\le x\le b}|f(x)-g(x)|, \quad f,g\in C[a,b].$$

We will provide a proof of Theorem 5 later in the course.

Equation (7) can be restated as $T(f)=f$, where

$$T(f)(x) = g(x) + \lambda \int_a^b k(x,y)f(y)\,dy. \tag{8}$$

Since g and k are both continuous, this defines an operator $T : C[a,b] \to C[a,b]$. Let us now determine for which values of λ the map T is a contraction. Note first that since k is continuous, it must also be bounded

$$|k(x,y)| \le c \quad \text{for all } (x,y) \in [a,b] \times [a,b]. \tag{9}$$

We have

$$\begin{aligned} d_\infty(T(f_1), T(f_2)) &= \max_{a \le x \le b} |T(f_1)(x) - T(f_2)(x)| \\ &= |\lambda| \max_{a \le x \le b} \left| \int_a^b k(x,y)\,(f_1(y) - f_2(y))\; dy \right| \\ &\le |\lambda| \max_{a \le x \le b} \int_a^b |k(x,y)||f_1(y) - f_2(y)|\, dy \\ &\le c|\lambda| \max_{a \le x \le b} |f_1(x) - f_2(x)| \int_a^b \; dy \\ &= c|\lambda|(b-a)d(f_1, f_2). \end{aligned}$$

Recall that T is a contraction if

$$d(T(f_1), T(f_2)) \le K d(f_1, f_2) \quad \text{for all } f_1, f_2 \in C[a,b]$$

for some constant $0 < K < 1$, and we see that this is indeed the case if

$$|\lambda| < \frac{1}{c(b-a)}. \tag{10}$$

In light of Theorem 5, Banach's Fixed Point Theorem now gives:

Theorem 6. Suppose k and g in (7) are continuous on $[a,b] \times [a,b]$ and $[a,b]$, respectively, and assume that the parameter λ satisfies (10), with c defined in (9). Then the integral equation (7) has a unique solution $f \in C[a,b]$. This solution is the limit of the iterative sequence $(f_0, f_1, f_2, \ldots)$, where f_0 is any continuous function on $[a,b]$, and

$$f_{n+1}(x) = g(x) + \lambda \int_a^b k(x,y) f_n(y)\, dy, \quad n = 1, 2, \ldots$$

1.2. **Application to differential equations.** Let us consider the *initial value problem*

$$x'(t) = \frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0, \tag{11}$$

where $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is a given function and $x(t)$ is an unknown function which we wish to determine. In this subsection we will use Banach's Fixed Point Theorem to prove the famous *Picard-Lindelöf Theorem*, which guarantees the uniqueness and existence of a solution to (11).

Theorem 7 (Picard-Lindelöf)**.** Let f be continuous on a rectangle

$$R = \{(t,x) : |t - t_0| \le a,\ |x - x_0| \le b\},$$

and thus bounded on R, say $|f(x,t)| \leq c$. Suppose that f satisfies a *Lipschitz condition* on R with respect to its second argument, meaning there exists a constant k such that

$$|f(t,x) - f(t,y)| \leq k|x-y| \quad \text{for all } (t,x),(t,y) \in R.$$

Then the initial value problem (11) has a unique solution which exists on an interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta < \min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}. \tag{12}$$

Proof. We split the proof into five steps.

Step 1: Equivalent formulation as an integral equation: We observe first that if a function $x \in C^1[t_0 - a, t_0 + a]$ solves (11), then necessarily

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{13}$$

by integration. On the other hand, if $x \in C[t_0 - a, t_0 + a]$ fulfils (13), then x is a continuously differentiable solution to (11) (this follows from the Fundamental Theorem of Calculus). Thus, the initial value problem (11) for $x \in C^1[t_0 - a, t_0 + a]$ is equivalent to (13) for $x \in C[t_0 - a, t_0 + a]$.

Step 2: Constructing an operator T on a complete space to which we can apply Banach's Fixed Point Theorem: For $J = [t_0 - \beta, t_0 + \beta]$ and $y \in C(J)$, define the operator

$$T(y)(t) := x_0 + \int_{t_0}^{t} f(s, y(s))\, ds, \quad t \in J.$$

Consider the set

$$X := \left\{ y \in C(J) \,:\, y(t_0) = x_0, \sup_{t \in J} |x_0 - y(t)| \leq c\beta \right\}.$$

This is a closed subspace of $C(J)$ (endowed with the metric d_∞), so (X, d_∞) is complete.

Step 3: Observe that $T : X \to X$: For $y \in X$, we need to show that $T(y) \in X$. Observe that $T(y)(t_0) = x_0$. Moreover, we have

$$|x_0 - T(y)(t)| = \left| \int_{t_0}^{t} f(s, y(s))\, ds \right| \leq |t - t_0| \cdot \max_{t \in J} |f(t, y(t))| \leq c\beta,$$

so $T(y) \in X$.

Step 4: Showing T is a contraction: Fix $y_1, y_2 \in X$. We have

$$\begin{aligned} |T(y_1)(t) - T(y_2)(t)| &= \left| \int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s))\, ds \right| \\ &\leq |t - t_0| \cdot \max_{s \in J} k|y_1(s) - y_2(s)| \\ &\leq k\beta d(y_1, y_2). \end{aligned}$$

The right hand side above is independent of t, so taking the maximum over $t \in J$ on both sides, we get

$$d(T(y_1), T(y_2)) \leq k\beta d(y_1, y_2).$$

Recalling (12), we see that $k\beta < 1$, so T is a contraction on X.

Step 5: Conclusion: Banach's Fixed Point Theorem implies that T has a unique fixed point $x \in X$ such that

$$x(t) = T(x)(t) = x_0 + \int_t^{t_0} f(s, x(s))\, ds.$$

It thus follows from Step 1 that (11) has a unique, continuous solution $x(t)$ on the interval $[t_0 - \beta, t_0 + \beta]$. □

In addition to existence and uniqueness of a solution, Banach's Fixed Point Theorem provides us with an iterative procedure for finding the solution.

Corollary 8 (Picard iteration)**.** Under the assumptions of the Picard-Lindelöf Theorem, the sequence given by

$$x_0(t) = x_0, \quad x_{n+1}(t) = T(x_n)(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds, \quad n = 1, 2, \ldots,$$

converges uniformly to the unique solution $x(t)$ on $J = [t_0 - \beta, t_0 + \beta]$.

Note, however, that the practical usefulness of Picard iteration is rather limited, due to the integrations involved. This is illustrated by the following example.

Example 9. The first Picard iteration for the initial value problem

$$x'(t) = \sqrt{x} + x^3, \quad x(1) = 2,$$

is given by

$$x_1(t) = 2 + \int_1^t \left(\sqrt{2} + 2^3\right)\, ds = 2 + \left(\sqrt{2} + 8\right)(t - 1).$$

The second is

$$\begin{aligned} x_2(t) &= 2 + \int_1^t \left(\sqrt{x_1(s)} + (x_1(s))^3\right)\, ds \\ &= 2 + \int_1^t \left(\sqrt{2 + \left(\sqrt{2} + 8\right)(s - 1)} + \left(2 + \left(\sqrt{2} + 8\right)(s - 1)\right)^3\right) ds. \end{aligned}$$

We see that the second integral obtained with Picard iteration looks quite uninviting. The next iterations $x_3, x_4, \ldots$ will involve even worse integrals, illustrating that Picard iteration is of limited value in this case.