Oppgave 1 Let $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$.

a) Find a singular value decomposition of A.

b) Find the pseudoinverse A^+ of A and use it to find the best approximation to a solution of the system of linear equations:

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + x_2 - x_3 = 1.$$

Solution:

a) We provide examples of matrices U, Σ, V such that $A = U\Sigma V^*$.

$$A^*A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad AA^* = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

singular values: $\sigma_1 = 2, \sigma_2 = \sqrt{2}$ and $\sigma_3 = 0$

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

Eigenvector v_1 of A^*A for eigenvalue 2: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Eigenvector v_2 of A^*A for eigenvalue 2: $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Eigenvector v_1 of A^*A for eigenvalue 0: $v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

b) Pseudoinverse $A^+ = V^*\Sigma^+ U$ is $\frac{1}{4}\begin{pmatrix}1 & -1\\ 1 & -1\\ 2 & 2\end{pmatrix}$ for $\Sigma^+ = \begin{pmatrix}1/2 & 0\\ 0 & 1/\sqrt{2}\\ 0 & 0\end{pmatrix}$ and hence $x^+ =$

$$A^+b = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Oppgave 2 Find real numbers a, b, c such that

$$\int_0^1 |\sin(\pi x) - a - bx - cx^2|^2 dx$$

is minimal.

Solution:

Equivalently, we have to find the best approximation of $f(x) = \sin(\pi x)$ from the closed subspace $M = \operatorname{span}\{1, x, x^2\}$. We use that this is equivalent to finding a $p \in M$ such that $\langle f - p, q\rangle = 0$ for all $q \in M$. Since $\{1, x, x^2\}$ is a basis we have to only check this for the monomials:

$$0 = \int_0^1 (\sin(\pi x) - a - bx - cx^2) \cdot 1\, dx$$
$$0 = \int_0^1 (\sin(\pi x) - a - bx - cx^2) \cdot x\, dx$$
$$0 = \int_0^1 (\sin(\pi x) - a - bx - cx^2) \cdot x^2\, dx$$

which gives us a system of equations for the coefficients a, b, c. Let us first compute $\int_0^1 x\sin(\pi x)dx$ and $\int_0^1 x\sin(\pi x)dx$ with partial integration.

$$\int_0^1 x\sin(\pi x)dx = \frac{\sin(\pi x) - \pi x\cos(\pi x)}{\pi^2}\Big|_0^\pi = \frac{1}{\pi}$$
$$\int_0^1 x^2\sin(\pi x)dx = \frac{2\pi x\sin(\pi x) + (2 - \pi^2 x^2)\cos(\pi x)}{\pi^3}\Big|_0^\pi = \frac{\pi^2 - 4}{\pi^3}$$

The coefficients have to satisfy

$$\frac{2}{\pi} = a + \frac{b}{2} + \frac{c}{3}$$
$$\frac{1}{\pi} = \frac{a}{2} + \frac{b}{3} + \frac{c}{4}$$
$$\frac{\pi^2 - 4}{\pi^3} = \frac{a}{3} + \frac{b}{4} + \frac{c}{5}$$

The solution is given by $a = \frac{12(\pi^2-10)}{\pi^3}$, $b = -\frac{60(\pi^2-12)}{\pi^3}$ and $c = \frac{60(\pi^2-12)}{\pi^3}$.

Oppgave 3 Determine whether the following statements are true or false. If the statement is true, no further explanation is required. In case the statement is false, give a counterexample.

1. Let $(X, \|.\|)$ be an infinite-dimensional normed space. Any linear operator $T : X \to X$ is continuous.
   No. For example, $T : \ell^2 \to \ell^2$ defined by $T(x_n) = (nx_n)$ is not continuous, since it is not a bounded operator.

2. Let (X, d) be a metric space. Then any contraction $T : X \to X$ has a fixed point.
   No. Let T(x)=ax+b for $0 < a < 1$ and $b \in \mathbb{R}$. We consider T as a mapping of $\mathbb{R}$ without the point $\{\frac{b}{1-a}\}$, i.e. $X = \mathbb{R}\backslash\{\frac{b}{1-a}\}$ and with $d(x,y) = |x-y|$. Note that $(X, |.|)$ is not complete. Then $T : X \to X$ is a contraction, but there is no $x \in X$ such that $T(x) = x$. it does not have a fixed point.

3. Any $n \times n$-matrix with real entries can be diagonalized.
   No. For example $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable.

4. Let $C[0,1]$ denote the space of real-valued continuous functions on the interval $[0,1]$. Then $(C[0,1], \|.\|_\infty)$ is a separable Banach space, where $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$ for $f \in C[0,1]$.
   Yes.

Oppgave 4 Let $(\ell^2, \langle ., . \rangle)$ be the Hilbert space of complex-valued sequences equipped with the inner product $\langle (x_k)_k, (y_k)_k \rangle = \sum_{k=0}^{\infty} x_k \overline{y_k}$. We define the operator $T_a : \ell^2 \to \ell^2$ by

$$T_a x = (a_k x_k)_{k\in\mathbb{N}},$$

where a is a complex-valued bounded sequence, $a = (a_k)_k \in \ell^\infty$.

1. Show that T_a is a bounded linear operator on ℓ^2, and determine the operator norm of T_a.

2. Determine for which $a \in \ell^\infty$ the operator T_a is (i) unitary, (ii) self-adjoint.

3. Find a non-zero sequence $a \in \ell^\infty$ such that T_a is not surjective.

Solution:

a) We have for all $x ß\ell^2$

$$\|T_a x\|_2 = \|(a_1 x_1, a_2 x_2, ...)\|_2 \leq \|a\|_\infty \|x\|_2$$

and thus T_a is bounded and $\|T_a\| \leq \|a\|_\infty$.
The elements of the standard basis $e_n = (0, 0, .., 1, 0, ...)$ are eigenvectors of T_a for the eigenvalue a_n: $T_a(e_1) = a_1 = (a_1\, 1, a_2\, 0, ...) = a_1 e_1$, etc. Consequently,

$$\|T_a\| = \sup_{\|x\|_2=1} \|T_a\|_2 \geq \sup_{n\in\mathbb{N}} \|T_a(e_n)\|_2 = \sup_{n\in\mathbb{N}} |a_n| = \|a\|_\infty.$$

b) First we determine the adjoint of T_a:

$$\begin{aligned}\langle T_a x, y\rangle &= \sum_{i=1}^{\infty} a_i x_i \overline{y_i} \\ &= \sum_{i=1}^{\infty} x_i \overline{\overline{a_i} y_i} \\ &= \langle x, T_{\overline{a}} y\rangle = \langle x, T_a^* y\rangle\end{aligned}$$

and therefore $T_a^* = T_{\overline{a}}$. T_a is selfadjoint if and only if $T_a^* = T_a$ which is the case when $a = \overline{a}$, i.e. $a = (a_i)$ with $a_i \in \mathbb{R}$ for all $i \in \mathbb{N}$.
T_a is selfadjoint if and only if $T_a^* T_a = T_a T_a^* = I$:

$$T_a^* T_a(x_i) = (\overline{a_i} a_i x_i) = (a_i \overline{a_i} x_i) = T_a T_a^*(x_i) = (x_i)$$

if and only if $a_i \overline{a_i} = 1$ for all $i \in \mathbb{N}$.

c) For example, $a = (0, 1, 1, 1, ...)$ or more generally any $a = (a_i)$ with $\inf_i a_i = 0$ would be valid choices.

Oppgave 5 Let $(X, \|.\|)$ be a normed space and let $T : X \to X$ be a linear operator.

a) Show that if T is continuous on X, then T is a bounded operator.

b) Show that if T is a bounded operator on $(X, \|.\|)$, then T is continuous on X.

Solution:

a) Assume that T is continuous, and consider an arbitrary point $x_0 \in X$. Then given any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\|Tx - Tx_0\| \leq \varepsilon \quad \text{whenever} \quad \|x - x_0\| \leq \delta.$$

Now take any $y \in X \setminus \{0\}$, and set

$$x = x_0 + \frac{\delta}{\|y\|}y. \quad \text{Then} \quad x - x_0 = \frac{\delta}{\|y\|}y,$$

and we have $\|x - x_0\| = \delta$. By the linearity of T we get

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T\left(\frac{\delta}{\|y\|}y\right)\right\| = \frac{\delta}{\|y\|}\|Ty\|,$$

and from the continuity of T it thus follows that

$$\frac{\delta}{\|y\|}\|Ty\| \leq \varepsilon \quad \Rightarrow \quad \|Ty\| \leq \frac{\varepsilon}{\delta}\|y\|.$$

Thus, T is bounded and $\|T\| \leq \varepsilon/\delta$.

b) For $T = 0$ the statement is trivial, so let $T \neq 0$ and thus $\|T\| \neq 0$. Assume first that T is bounded. Consider any $x_0 \in X$ and any fixed $\varepsilon > 0$. Then, since T is linear, for every $x \in X$ with

$$\|x - x_0\| < \delta = \frac{\varepsilon}{\|T\|},$$

we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\|\|x - x_0\| < \|T\|\delta = \varepsilon.$$

This shows that T is continuous at x_0, which was chosen arbitrarily, so T is continuous.

Oppgave 6 Let $(X, \|.\|)$ be a normed space and T a bounded, invertible linear operator on X such that the inverse of T is also bounded on X. We set $\|x\|_T := \|Tx\|$.

a) Show that $\|.\|_T$ defines a norm on X.

b) Show that the two norms $\|.\|$ and $\|.\|_T$ are equivalent on X.

Solution:

a) Positivity: $\|x\|_T = \|Tx\| \geq 0$ since $\|.\|$ is a norm on X.
Homogenity: $\|\alpha\, x\|_T = \|T(\alpha\, x)\| = \|\alpha\, T(x)\| = |\alpha|\|Tx\| = |\alpha|\|x\|_T$ for $\alpha \in \mathbb{F}$, where we used the homogenity of $\|.\|$ and that T is a linear map on X.
Triangle inequality: By the linearity of T and the triangle inequality for $\|.\|$:

$$\begin{aligned} \|x+y\|_T &= \|T(x+y)\| = \|Tx+Ty\| \\ &\leq \|Tx\| + \|Ty\| = \|x\|_T + \|y\|_T. \end{aligned}$$

b) Claim: There exist two finite positive constants C_1, C_2 such that

$$C_1\|x\| \leq \|x\|_T \leq C_2\|x\| \qquad \text{for all } \ x \in X.$$

Since T is bounded we have that $\|Tx\| \leq \|T\|\|x\|$. Hence we have that

$$\|Tx\| \leq C_2\|x\|$$

holds for any $C_2 \geq \|T\|$.

The existence of C_2 relies on the invertibility of T and that $\|T^{-1}\|$ is finite:

$$\|x\| = \|T^{-1}(Tx)\| \leq \|T^{-1}\|\|Tx\|$$

and thus we have for any C_1 with $C_1 \leq \|T^{-1}\|^{-1}$ that

$$C_1\|x\| \leq \|x\|_T \qquad \text{for all } \ x \in X.$$

In summary, we have the optimal constants:

$$\|T^{-1}\|^{-1}\|x\| \leq \|x\|_T \leq \|T\|\|x\| \qquad \text{for all } \ x \in X.$$