$$\Sigma = \begin{pmatrix} 6\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

(·) $U = (u_1 | u_2 | u_3)$ is found using

$$u_i = \frac{1}{\sigma_i} A v_i \quad , i = 1, 2.$$

$$u_1 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \quad u_2 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Now extend $\{u_1, u_2\}$ to a basis for $\mathbb{R}^3$. In this case, one choice is

$$u_3 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

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2. $A^+ = V \Sigma^+ U^*$

$$= \frac{1}{36}\begin{pmatrix} 5 & -4 & -2 \\ -3 & 0 & 6 \end{pmatrix}.$$

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3. $Ax = b$ for $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$x^+ = A^+ b = \frac{1}{12}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ is minimal

Problem 2

a) ③ ; b) ① ; c) ③

Problem 3

1. No. The set of binary sequences
$B = \{(x_0)_n : x_n \in \{0,1\}, n \in \mathbb{N}\}$
is uncountable and thus cannot be covered by countably many balls of radius less than 1.

2. No. There are bounded lin. operators with non-closed range.
Example: $T: l^\infty \to l^\infty$, multiplication operator $T:(x_n) \mapsto \left(\frac{x_n}{n}\right)$.

3. No. There are non-diagonalizable matrices, such as $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ wrt stand. basis of $\mathbb{R}^3$.

4. Yes. A separable Hilbert space has an orthonormal basis.

5. No.

Problem 4

1. If T is invertible, then there exists a bounded op. T^{-1}, such that $T^{-1}T = TT^{-1} = I$ on X.

Hence $\|T^{-1}T\| = \|I\| = 1$, i.e.

$$\|T^{-1}(Tx)\| = \|x\| \text{ for all } x \in X.$$

Since $\|T^{-1}(Tx)\| \leq \|Tx\| \, \|T^{-1}\|$,

we obtain that

$$\frac{1}{\|T^{-1}\|}\|x\| \leq \|Tx\| \text{ for all } x \in X.$$

Thus $\|Tx\| \geq C\|x\|$ for $C = \frac{1}{\|T^{-1}\|}$.

2. Since there exists a $C > 0$ s.t.

$$\|Tx\| \geq C\|x\| \text{ for } x \in X,$$

we can argue as follows:

If $(Tx_n)_n$ is a Cauchy sequence in $\operatorname{ran}(T)$, then

$\|x_n - x_m\| \leq \frac{1}{C}\|Tx_n - Tx_m\|$ implies that $(x_n)_n$ is a Cauchy seq. in X.

Hence there exists an $x \in X$ s.t.

$$x = \lim_n x_n . \quad (*)$$

Consequently, $Tx = \lim_n Tx_n$ is in $\operatorname{ran}(T)$ and thus $\operatorname{ran}(T)$ is a closed subspace of X.

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3. Since $\operatorname{ran} T$ is dense in X and note that $\|Tx\| \geq C\|x\|$ for all $x \in X$ implies that T is injective on X, we deduce that the inverse of T, T^{-1}, is well-defined on X.

If $y = Tx$, then

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{C}\|Tf\| = \frac{1}{C}\|y\|$$

$\Rightarrow T^{-1}$ is bounded. $\square$

$(*)$ $\|Tx - Tx_n\| \leq \|T\| \|x - x_n\|$

Problem 5

$$3x_1 + \frac{3}{2}x_2 - x_3 = 1$$
$$-x_1 + 4x_2 - \frac{4}{5}x_3 = 2$$
$$\frac{2}{5}x_1 - \frac{1}{4}x_2 + 2x_3 = 4,$$

is equivalent to

$$x_1 = \frac{x_2}{2} + \frac{x_3}{3} + \frac{2}{3}$$
$$x_2 = \frac{x_1}{4} + \frac{x_3}{5} + \frac{1}{2}$$
$$x_3 = -\frac{x_1}{5} + \frac{x_2}{8} + 2.$$

Hence we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{8} & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 2 \end{pmatrix}.$$

Let us view T as a mapping from

$$(\mathbb{R}^3, \|\cdot\|_\infty) \text{ to } (\mathbb{R}^3, \|\cdot\|_\infty).$$

$\|Ax\|_\infty \leq \|A\|_\infty \|x\|_\infty$, where

$$\|A\|_\infty = \sup_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_i \sum_{j=1}^{3} |a_{ij}|$$

$$= \max_i \left\{ \frac{1}{4} + \frac{1}{5} \,|\, \frac{1}{2} + \frac{1}{8} \,|\, \frac{1}{3} + \frac{1}{5} \right\} < 1.$$

Hence, $Tx = Ax + \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 2 \end{pmatrix}$ is

a contraction wrt $\|\cdot\|_\infty$. $= c$

Since $(\mathbb{R}^3, \|\cdot\|_\infty)$ is a Banach space, there exists a unique solution to $x = Ax + c$ by Banach's fixed point theorem.

This solution may be found as the limit of the sequence $(x_n)_n$, where $x_{n+1} = Ax_n + c$ for $n \geq 0$ and $x_0 \in \mathbb{R}^3$.

$$x^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x^{(1)} = Ax^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{8} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{8} \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} \frac{1}{24} \\ \frac{3}{20} \\ -\frac{1}{10} \end{pmatrix}.$$

Problem 1

$$A = \begin{pmatrix} 4 & 0 \\ -5 & 3 \\ 2 & 6 \end{pmatrix}$$

1. Find a SVD of A:

$$A = \underbrace{\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 6\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{V^*}$$

Solution:

$$A^*A = \begin{pmatrix} 45 & 27 \\ 27 & 45 \end{pmatrix} \rightsquigarrow \sigma_1^2 = 18, \sigma_2^2 = 72$$

$$\Rightarrow \sigma_1 = 3\sqrt{2}$$

$$\sigma_2 = 6\sqrt{2}$$

$\rightsquigarrow$ Singular vector for σ_1: $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Singular vector for σ_2: $v_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \end{pmatrix}$