



Please justify your answers! The most important part is how you arrive at an answer, not the answer itself.

- 1 Show that point evaluation is a bounded linear functional on $C[a, b]$. That is, for some fixed $t_0 \in [a, b]$ define $f_{t_0} : C[a, b] \rightarrow \mathbb{C}$ by

$$f_{t_0}(x) = x(t_0), \quad x \in C[a, b],$$

and show that f_{t_0} is a bounded linear functional on $C[a, b]$.

- 2 Let T be a bounded linear operator on a real Hilbert space X . Show that the operator norm of T can be expressed in terms of the inner product of X :

$$\|T\| = \sup\{\langle Tx, y \rangle : x, y \in X \text{ with } \|x\| = \|y\| = 1\}.$$

- 3 Let c_f be the subspace of ℓ^2 that consists of all sequences with finitely many non-zero terms.

- a) Show that best approximation fails for c_f .
b) Why does this not contradict the best approximation theorem?

- 4 Let M be a subspace of an inner product space $(X, \langle \cdot, \cdot \rangle)$. Show that the orthogonal complement M^\perp is closed.

- 5 Let M be the plane of \mathbb{R}^3 given by $x_1 + x_2 + x_3 = 0$. Find the linear mapping that is the orthogonal projection of \mathbb{R}^3 onto this plane.

- 6 (Exam 2017, problem 4) For $a = (a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{R})$ we define the linear operator $T_a : \ell^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{R})$ by

$$T_a(x_1, x_2, \dots) = (a_1x_1, 0, a_3x_3, 0, \dots), \quad x \in \ell^2(\mathbb{R}).$$

- a) Show that T_a is bounded on $\ell^2(\mathbb{R})$.
- b) Determine the operator norm of T_a .
- c) Show that the range of T_a is closed.
- d) Determine the orthogonal complement of $\ker(T_a)$.
- e) Determine for which sequences $a \in \ell^\infty(\mathbb{R})$ the operator T_a satisfies $T_a^2 = T_a$.