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Exercise set 8

1 Which of the following transformations are linear?

a) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p)(x) = xp(x) + p'(x)$, where $P_n(\mathbb{R})$ denotes the vector space of real-valued polynomials of degree at most n.

b) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = (\overline{z_1}, \overline{z_2})$, where $\mathbb{C}^2$ is a vector space over $\mathbb{R}$.

Does the conclusion change if $\mathbb{C}^2$ is considered as a vector space over $\mathbb{C}$? Explain.

c) Let $M_{n\times n}(\mathbb{R})$ denote the space of all $n \times n$ matrices with real entries.

i) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, $T(A) = A^2$.

ii) $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $T(A) = \det A$.

2 Let X and Y be normed spaces. Show that a linear map $T : X \to Y$ is not continuous if and only if there exists a sequence of unit vectors (x_n) in X such that $\|Tx_n\| \geq n$ for $n \in \mathbb{N}$.

3 Let X and Y be vector spaces, both real or both complex. Let $T : X \to Y$ be a linear operator with some range $\operatorname{ran}(T) \subset Y$. Show that:

a) The inverse operator $T^{-1} : \operatorname{ran}(T) \to X$ exists if and only if

$$Tx = 0 \quad \Rightarrow \quad x = 0.$$

(In other words: if and only if $\ker(T) = \{0\}$.)

b) If T^{-1} exists, it is a linear operator.

c) Even if T is a bounded operator, its inverse T^{-1} need not be.

Note: The inverse operator $T^{-1} : \operatorname{ran}(T) \to X$ is an operator satisfying $T^{-1}(T(x)) = x$ and $T(T^{-1}(y))$ for any $x \in X$ and $y \in \operatorname{ran}(T)$.

4 Let T be a linear mapping $T : (\mathbb{R}^n, \|.\|_\infty) \to (\mathbb{R}^n, \|.\|_\infty)$ given by a $n \times n$ matrix A. Show that the operator norm of T in terms of A is given by $\|T\| = \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|$.

5 Let T be the integral operator $Tf(x) = \int_0^1 k(x,y)f(y)dy$ defined by a kernel $k \in C([0,1] \times [0,1])$ such that $k(x,y) \geq 0$ for any $(x,y) \in [0,1] \times [0,1]$. Show that the operator norm of T as a mapping on $C[0,1]$ with respect to $\|.\|_\infty$-norm is $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$.