Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2019
Exercise set 7

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let c_f denote the space of real-valued sequences with only finitely many non-zero entries. Show that c_f is dense in $\ell^p(\mathbb{R})$ for any $1 \leq p < \infty$.

2

a) Illustrate with an example that in Banach's fixed point theorem, completeness of the space is essential and cannot be omitted.

b) It is also essential that T is a *contraction*; it is not enough that

$$d(Tx, Ty) < d(x, y) \quad \text{when } x \neq y.$$

To see this, consider $X = [1, \infty) \subset \mathbb{R}$ taken with the usual $|\cdot|$ norm, and

$$T : X \to X \quad \text{defined by } x \to x + \frac{1}{x}.$$

Show that $|Tx - Ty| < |x - y|$ when $x \neq y$, but the mapping has no fixed points.

3 Problem 1, exam 2007:

Let $G : C[0,1] \to C[0,1]$ be defined by

$$(Gx)(t) = \int_0^t s x(s)\, ds, \quad 0 \leq t \leq 1.$$

a) Show that G is a contraction if $C[0,1]$ has the $\|\cdot\|_\infty$-norm.

b) Define $F : C[0,1] \to C[0,1]$ by

$$(Fx)(t) = \frac{t^2}{2} - (Gx)(t), \quad 0 \leq t \leq 1.$$

Show that if $x_0(t) = 0$ for all t, then

$$(F^n x_0)(t) = \sum_{k=1}^{n} (-1)^{k+1} \frac{t^{2k}}{2^k k!}, \quad n = 1, 2, \ldots$$

Hint: Induction.

c) Explain why F has a unique fixed point x^*, and find x^* by iteration.

4 Consider the integral equation

$$f(x) = \sin x + \lambda \int_0^3 e^{-(x-y)} f(y) dy$$

for some scalar λ.

a) Determine for which λ there exists a continuous function f on $[0, 3]$ that solves this integral equation.

b) Pick one of the values of λ found in **a)**. Use the method of iteration, as described in Banach's fixed point theorem, to find approximations f_1 and f_2 to a potential solution by starting with $f_0(x) = 1$ on $[0, 3]$.

5 Apply Picard iteration to

$$x'(t) = 1 + x^2, \quad x(0) = 0.$$

Find x_3 and the exact solution (notice that the equation is separable), and show that the terms involving $t, t^2, \ldots, t^5$ in $x_3(t)$ are the same as those of the Taylor series of the exact solution.