Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2019

Exercise set 6

1 Let (X,d) be a metric space. Show that any convergent sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in X.

2 Let (X,d) be a metric space, and assume that $Y \subset X$ is a dense subset of X. Show that for any $x \in X$ there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $x = \lim_{n\to\infty} y_n$.

3 Prove the following two statements for a normed space $(X, \|.\|)$.

a) Any ball $B_r(x) = \{y \in X : \|x-y\| < r\}$ in $(X,\|.\|)$ is bounded and $\operatorname{diam}(B_r(x)) \leq 2r$.

b) If A is a bounded subset of $(X,\|.\|)$, then for any $a \in A$ we have $A \subseteq \bar{B}_{\operatorname{diam}(A)}(a)$. (Recall that the a closed ball $\bar{B}_r(x)$ is the set $\{y \in X : \|y-x\| \leq r\}$.)

4 **a)** Let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(t) = \begin{cases} 0 & \text{for } a \leq t \leq \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \leq \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \leq t \leq b. \end{cases}$$

in $C[a,b]$. Use the definition of uniform convergence to determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$.

b) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence on $[0,1]$ defined by $f_n(x) = \frac{1}{1+nx}$. Use the definition of uniform convergence to determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$.

5 Show that $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

6 Let c_0 denote the space of real-valued sequences converging to zero.

a) Show that $(c_0, \|\cdot\|_\infty)$ is a subspace of $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$.

b) Show that c_0 is closed in $\ell^\infty(\mathbb{R})$, and conclude that $(c_0, \|\cdot\|_\infty)$ is complete.