



Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

**1** Let  $(x_n)_{n \in \mathbb{N}}$  be a convergent sequence in a metric space  $(X, d)$ .

**a)** Show that  $(x_n)_{n \in \mathbb{N}}$  is a bounded subset of  $X$ .

**b)** Show that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence.

**2** Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space, and suppose that  $(x_n)$  and  $(y_n)$  are convergent sequences with  $\lim_n x_n = x$  and  $\lim_n y_n = y$ . Show that

$$\lim_{n \rightarrow \infty} \langle x_n, y_n \rangle = \langle x, y \rangle.$$

(This was exam problem 5b in Spring 2015.)

**3** We denote by  $c_f$  the vector space of all sequences with only finitely many non-zero terms. Show that  $c_f$  is not a Banach space with the norm  $\|\cdot\|_\infty$ . As usual,  $\|\cdot\|_\infty$  is defined by

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

for a sequence  $x = (x_n)_{n \in \mathbb{N}} \in c_f$ .

**4** For each  $n \in \mathbb{N}$ , let

$$x^{(n)} := \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right),$$

which we regard as an element of the space  $\ell^p(\mathbb{R})$  (for any given  $p \in [1, \infty]$ ).

**a)** Find the limit of the sequence  $(x^{(n)})_{n \geq 1}$  in  $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$ . Prove your claim.

**b)** Does  $(x^{(n)})_{n \geq 1}$  have a limit in  $(\ell^1(\mathbb{R}), \|\cdot\|_1)$ ? If the limit exists, find it and prove that it is the limit.

- c) Does  $(x^{(n)})_{n \geq 1}$  have a limit in  $(\ell^2(\mathbb{R}), \|\cdot\|_2)$ ? If the limit exists, find it and prove that it is the limit.

**5** Let  $C[a, b]$  be the vector space of all continuous functions  $f: [a, b] \rightarrow \mathbb{R}$ . We will consider two norms on this space,  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$ .

- a) Prove that for all  $f \in C[a, b]$  we have

$$\|f\|_1 \leq (b - a) \|f\|_\infty.$$

- b) Let  $(f_n)$  be a sequence in  $C[a, b]$ .

Prove that if  $f_n \rightarrow f$  with respect to  $\|\cdot\|_\infty$  then  $f_n \rightarrow f$  with respect to  $\|\cdot\|_1$ .

- c) Show that the reverse of the statement in b) is not always true.