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TMA4145 Linear Methods

Fall 2019

Exercise set 4

1 Recall that $\mathcal{P}_4$ is the space of polynomials of degree at most 4. Show that the sets $U, V \subset \mathcal{P}_4$, defined by

$$U := \{p \in \mathcal{P}_4 : \ p(-1) = p(1) = 0\},$$
$$V := \{p \in \mathcal{P}_4 : \ p(1) = p(2) = p(3) = 0\}$$

are subspaces of $\mathcal{P}_4$ and determine the subspace $U \cap V$.

2 Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries. For $A \in M_n(\mathbb{C})$ we define its *trace* by $\operatorname{tr}(A) = a_{11} + \cdots + a_{nn}$.

a) Show that for $A, B \in M_3(\mathbb{C})$ we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and try to show this property of the trace for $n \times n$ matrices.

b) Let $S \subset M_n(\mathbb{C})$ be defined as the matrices with $\operatorname{tr}(A) = 0$. Show that S is a subspace of $M_n(\mathbb{C})$.

3 **a)** Prove that $(l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space, where for any bounded sequence $x = (x_n) \in l^\infty(\mathbb{R})$ we define

$$\|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n|.$$

Is this norm associated with an inner product?
Note: For the first part, you don't need to show that ℓ^∞ is a vector space, just that the axioms for a normed space are satisfied.

b) Show that the norm $\|\cdot\|_p$ on $\ell^p(\mathbb{R})$ does not satisfy the parallelogram law

$$\|x - y\|_p^2 + \|x + y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2 \quad \text{for all} \quad x, y \in X,$$

for any $p \neq 2$.

4 Find a sequence $x = (x_1, x_2, \dots)$ of real numbers which converges to 0, but which is not in any space $\ell^p(\mathbb{R})$, $1 \leq p < \infty$.

5 Suppose $(X, \langle .,. \rangle)$ is an inner product space, and let $\| \cdot \| = \langle \cdot \cdot \rangle^{1/2}$.

a) Show that $\| \cdot \|$ satisfies the parallelogram law.

b) Let ω be a n^{th} root of unity, i.e. $\omega^n = 1$ and $\omega^k \neq 1$ for $k < n$. Show that for $n \geq 3$

$$\langle x, y \rangle = \frac{1}{n} \sum_{k=1}^{n} \omega^k \|x + \omega^k y\|^2.$$

c) Show that

$$\langle x, y \rangle = \int_0^1 e^{2\pi i \varphi} \|x + e^{2\pi i \varphi} y\|^2 d\varphi.$$

6 Let $(\mathbb{R}^n, \|.\|_p)$ be the space of real n-tuples with p-norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for some $1 \leq p < \infty$. Show that

$$\sum_{i=1}^{n} |x_i| \leq n^{(p-1)/p} (\sum_{i=1}^{n} |x_i|^p)^{1/p}.$$