



- 1** a) Determine if the following expressions are norms for \mathbb{R}^3 .
1. $f(x_1, x_2, x_3) = |x_1| + |x_2|$;
 2. $f(x_1, x_2, x_3) = |x_1| + (|x_2|^2 + |x_3|^2)^{1/2}$;
 3. $f(x_1, x_2, x_3) = (w_1|x_1|^3 + w_2|x_2|^2 + w_3|x_3|)^{1/2}$ for some positive real numbers w_1, w_2, w_3 .
- b) Determine $\|z\|_1, \|z\|_2$ and $\|z\|_\infty$ for $z = (1+i, 1-i)$ and $z = (e^{i\pi/2}, e^{3i\pi/2})$ in \mathbb{C}^2 .
- 2** For $1 \leq p_1 < p_2 \leq \infty$ show that the sequence space ℓ^{p_1} is a proper subspace of ℓ^{p_2} .
- 3** Let X be a vector space and $\|\cdot\|_a$ and $\|\cdot\|_b$ norms on x . Show that $\|x\| := (\|x\|_a^2 + \|x\|_b^2)^{1/2}$ defines a norm on X .
- Challenge (voluntary): Try to define a variant of this norm for $p \neq 2$ and contemplate about a possible proof of this statement.*
- 4** Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices. Define for $A \in M_n(\mathbb{R})$ the function $\|A\|_2 = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$. Show that $\|\cdot\|_2$ is a norm on $M_n(\mathbb{R})$. The trace of a matrix $A \in M_n(\mathbb{R})$ is defined as the sum of its diagonal elements, $\text{tr}(A) = a_{11} + \dots + a_{nn}$. Prove that $\|A\|_2^2 = \text{tr}(A^T A)$. If the general case is too difficult, try to do it for $n = 3$.
- 5** Let $(X, \|\cdot\|)$ be a normed vector space. Show that for any $x, y \in X$ we have

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

This is known as the *reversed triangle inequality*.