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# TMA4145 Linear Methods
Fall 2019

**Exercise set 12**

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

**1** Let

$$z_1 = \sqrt{\frac{2}{3}}\begin{bmatrix}1\\0\end{bmatrix}, \quad z_2 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\ \sqrt{3}/2\end{bmatrix}, \quad z_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\ -\sqrt{3}/2\end{bmatrix}.$$

Show that for every $x \in \mathbb{R}^2$ we have

**a)**

$$\|x\|^2 = \sum_{i=1}^{3} |\langle x, z_i\rangle|^2$$

**b)**

$$x = \sum_{i=1}^{3} \langle x, z_i\rangle z_i$$

*Remark.* The vectors $z_1, z_2, z_3$ span $\mathbb{R}^2$, but they are obviously not an orthonormal basis (they are not even linearly independent). Still, they satisfy a generalization of Parseval's identity and "act like" an orthonormal basis. Such systems appear very naturally in applications (e.g. in signal analysis), and are often called Parseval frames.

**2** Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent norms on a vector space $X$. Show that any set $U \subset X$ is open in $(X, \|\cdot\|_a)$ if and only if it is open in $(X, \|\cdot\|_b)$.

*Remark.* This is in fact a two-way implication; if any set $U \subset X$ is open in $(X, \|\cdot\|_a)$ if and only if it is open in $(X, \|\cdot\|_b)$, then necessarily the norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent on $X$.

**3** Suppose that $v_1, \ldots, v_k$ are non-zero eigenvectors of an operator $T$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Show that $\{v_1, \ldots v_k\}$ is a linearly independent set.

**4** Let $T$ be the shift operator on $\ell^2$ defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$.

1. Show that $T$ has no eigenvalues.
2. Does $T^*$ have any eigenvalues?

**5** Let $U$ be a $n \times n$ matrix with columns $u_1, ..., u_n$. Show that the following statements are equivalent:

1. $U$ is unitary.
2. $\{u_1, ..., u_n\}$ is an orthonormal basis of $\mathbb{C}^n$.

**6** Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

**a)** Compute the singular value decomposition of $A$.

**b)** Use the result of a) to find:

1. The pseudo-inverse of $A$.
2. The minimal norm solution of $Ax = b$ for $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.