



Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

- 1 Let X be a Hilbert space and suppose x and x' are two elements in X . Show that if

$$\langle x, y \rangle = \langle x', y \rangle \quad \text{for all } y \in X,$$

then $x = x'$.

- 2 Define on $C[0, 1]$ the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)} dt$$

Show that $(C[0, 1], \langle \cdot, \cdot \rangle)$ is an inner product space, but that it is not complete with respect to the norm

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt \right)^{1/2}$$

induced by the inner product.

- 3 Let X_1 and X_2 be two Hilbert spaces and $T \in B(X_1, X_2)$.

a) Show that there exists $T^* \in B(X_2, X_1)$ such that $\langle Tx, y \rangle_{X_2} = \langle x, T^*y \rangle_{X_1}$ for any $x \in X_1, y \in X_2$.

(Note: We treated the case $X_1 = X_2$ in class.)

b) Prove that $\ker T = \ker T^*T$.

- 4 Let $T : X \rightarrow X$ be a bounded linear operator on a Hilbert space X . Show that

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

- 5 Consider the multiplication operator T_a on $(\ell^2, \langle \cdot, \cdot \rangle)$ given by

$$T_a x = (a_j x_j)_{j \in \mathbb{N}}$$

for a fixed sequence $a = (a_j)_{j \in \mathbb{N}} \in \ell^\infty$.

- a) Determine the adjoint operator T_a^* .
- b) Is T_a a normal operator? Under which condition(s) on the sequence a is T_a unitary; self-adjoint?
- 6 Let M be a closed subspace of a Hilbert space X , which by the projection theorem is given by the direct sum

$$X = M \oplus M^\perp.$$

Show that the projection onto M is self-adjoint.