Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 7:Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let c_f denote the space of real-valued sequences with only finitely many non-zero entries. Show that c_f is dense in $\ell^p(\mathbb{R})$ for any $1 \leq p < \infty$.

Solution. From the comment after the definition of dense subsets in the notes, we know how to show that c_f is a dense subset of ℓ^p: for every $x \in \ell^p$ and every $\epsilon > 0$, we need to find some $y \in c_f$ such that $d(x,y) = \|x-y\|_{\ell^p} < \epsilon$. Assume that $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$. It is intuitively clear that if we want to approximate $(x_n)_{n\in\mathbb{N}}$ by a finite sequence, this finite sequence should consist of the first N elements of $(x_n)_{n\in\mathbb{N}}$. We need to find an $N \in \mathbb{N}$ that works.
By definition of the space ℓ^p, we know that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

As you know from earlier calculus classes, the tail of a convergent series approaches zero, i.e.

$$\sum_{n=k}^{\infty} |x_n|^p \to 0 \text{ as } k \to \infty.$$

This means that we can find some $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |x_n|^p < \epsilon^p$. Now let $y \in c_f$ be the sequence

$$y = (x_1, x_2, \dots x_{N-1}, x_N, 0, 0, \dots).$$

In words, y consists of the first N elements of x, and all the other elements are zero.

Then

$$\begin{aligned}\|x-y\|_{\ell^p} &= \left(\sum_{n=1}^{\infty}|x_n-y_n|^p\right)^{1/p}\\ &= \left(\sum_{n=1}^{N}|x_n-x_n|^p+\sum_{n=N+1}^{\infty}|x_n-0|^p\right)^{1/p}\\ &= \left(\sum_{n=N+1}^{\infty}|x_n|^p\right)^{1/p}\\ &< (\epsilon^p)^{1/p}=\epsilon.\end{aligned}$$

2 **a)** Illustrate with an example that in Banach's fixed point theorem, completeness of the space is essential and cannot be omitted.

b) It is also essential that T is a *contraction*; it is not enough that

$$d(Tx,Ty)<d(x,y) \quad \text{when } x\neq y.$$

To see this, consider $X=[1,\infty)\subset\mathbb{R}$ taken with the usual $|\cdot|$ norm, and

$$T\,:\,X\to X \quad \text{defined by } x\to x+\frac{1}{x}.$$

Show that $|Tx-Ty|<|x-y|$ when $x\neq y$, but the mapping has no fixed points.

Solution. a) Let $X=\mathbb{R}\setminus\{0\}$. Then X is not complete with respect to the usual norm given by the absolute value[1]. Consider the mapping $T:X\to X$ given by $Tx=\frac{x}{2}$. Then X is a contraction:

$$\begin{aligned}|Tx-Ty| &= \left|\frac{x}{2}-\frac{y}{2}\right|\\ &= \frac{1}{2}|x-y|;\end{aligned}$$

in the definition of contraction in the lecture notes, we can pick K to be any number in the open interval $(1/2,1)$. However, T has no fixed point. A fixed point would satisfy $Tx=x$, which means that

$$\frac{x}{2}=x.$$

[1]One way to see this is to note that X clearly is not a *closed* subset of the Banach space $\mathbb{R}$. Hence theorem 3.12 says that X is not complete.

This happens if and only if $x = 0$, but 0 is not an element of X.
b) Introducing the common denominator xy, we find for $x \neq y$ that

$$\begin{aligned}|Tx - Ty| &= \left| x + \frac{1}{x} - \left(y + \frac{1}{y} \right) \right| \\ &= \left| \frac{x^2y + y - y^2x - x}{xy} \right| \\ &= \left| \frac{x(xy-1) - y(xy-1)}{xy} \right| \\ &= \left| \frac{(x-y)(xy-1)}{xy} \right| \\ &= |x-y| \frac{|xy-1|}{|xy|} \\ &< |x-y|.\end{aligned}$$

To justify the last step, note that $x, y \in [1, \infty)$, so clearly $xy > 1$ whenever $x \neq y$. Therefore $|xy - 1| > |xy|$.
If T had a fixed point x_0, x_0 would satisfy

$$x_0 = x_0 + \frac{1}{x_0},$$

which clearly implies $\frac{1}{x_0} = 0$, which is not possible.

3 Problem 1, exam 2007:

Let $G : C[0,1] \to C[0,1]$ be defined by

$$(Gx)(t) = \int_0^t sx(s)\,ds, \quad 0 \leq t \leq 1.$$

a) Show that G is a contraction if $C[0,1]$ has the $\|\cdot\|_\infty$-norm.

b) Define $F : C[0,1] \to C[0,1]$ by

$$(Fx)(t) = \frac{t^2}{2} - (Gx)(t), \quad 0 \leq t \leq 1.$$

Show that if $x_0(t) = 0$ for all t, then

$$(F^n x_0)(t) = \sum_{k=1}^{n} (-1)^{k+1} \frac{t^{2k}}{2^k k!}, \quad n = 1, 2, \ldots$$

Hint: Induction.

c) Explain why F has a unique fixed point x^*, and find x^* by iteration.

Solution. a) To show that G is a contraction, we need to show that there is some $K \in (0,1)$ such that[2]

$$\|Gx - Gy\|_\infty \leq K\|x-y\|_\infty.$$

For some $x, y \in C[0,1]$ and $t \in [0,1]$, we find that

$$\begin{aligned} |Gx(t) - Gy(t)| &= \left| \int_0^t sx(s)\ ds - \int_0^t sx(s)\ ds \right| \\ &= \left| \int_0^t s\left(x(s) - y(s)\right)\ ds \right| \\ &\leq \int_0^t s|x(s) - y(s)|\ ds \end{aligned}$$

Now recall that by definition $\|x-y\|_\infty = \sup_{s\in[0,1]} |x(s) - y(s)|$. Since the supremum is an upper bound, we get that

$$|x(s) - y(s)| \leq \|x-y\|_\infty \text{ for any } s \in [0,1].$$

We insert this into our calculation:

$$\begin{aligned} \int_0^t s|x(s) - y(s)|\ ds &\leq \|x-y\|_\infty \int_0^t s\ ds \\ &= \|x-y\|_\infty \frac{t^2}{2} \\ &\leq \frac{1}{2}\|x-y\|_\infty \text{ for } t \in [0,1]. \end{aligned}$$

In total, we have shown that

$$|Gx(t) - Gy(t)| \leq \frac{1}{2}\|x-y\|_\infty \text{ for } t \in [0,1],$$

which implies that

$$\|Gx - Gy\|_\infty \leq \frac{1}{2}\|x-y\|_\infty$$

by the definition of the norm $\|\cdot\|_\infty$ as a supremum. If we pick any $K \in (1/2, 1)$, we have $0 < K < 1$ and

$$\|Gx - Gy\|_\infty < K\|x-y\|_\infty,$$

hence G is a contraction.

b) By writing out the definition of G, we see that

$$(Fx)(t) = \frac{t^2}{2} - \int_0^t sx(s)\ ds.$$

We want to show that

$$(F^n x_0)(t) = \sum_{k=1}^{n} (-1)^{k+1} \frac{t^{2k}}{2^k k!}, \quad n = 1, 2, \ldots. \tag{1}$$

[2] Recall that the distance on a normed space is given by $d(x,y) = \|x-y\|$.

As the hint suggests, we proceed using induction. For the base case $n = 1$, the left hand side of (1) is

$$(Fx_0)(t) = \frac{t^2}{2} - \int_0^t 0\ ds = \frac{t^2}{2}.$$

The right hand side of (1) is, for $n = 1$,

$$\sum_{k=1}^{1} (-1)^{k+1} \frac{t^{2k}}{2^k k!} = (-1)^{1+1} \frac{t^2}{2} = \frac{t^2}{2}.$$

Hence (1) is true for $n = 1$. For the induction step, assume that (1) holds for $n = m$ – we need to show that (1) then holds for $n = m + 1$. We will start with the left hand side of (1) for $n = m + 1$, and use the induction hypothesis to manipulate it into the right hand side of the equation.

$$\begin{aligned}
(F^{m+1}x_0)(t) &= (F(F^m x_0))(t) \\
&= \frac{t^2}{2} - \int_0^t s(F^m x_0)(s)\ ds \\
&= \frac{t^2}{2} - \int_0^t s \sum_{k=1}^{m} (-1)^{k+1} \frac{s^{2k}}{2^k k!}\ ds && \text{(induction hypothesis)} \\
&= \frac{t^2}{2} - \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^k k!} \int_0^t s^{2k+1}\ ds \\
&= \frac{t^2}{2} - \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^k k!} \frac{t^{2k+2}}{2k+2} \\
&= \frac{t^2}{2} - \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^{k+1}(k+1)!} t^{2k+2} && \text{(using } 2k+2 = 2(k+1)\text{)} \\
&= \frac{t^2}{2} + \sum_{k=1}^{m} \frac{(-1)^{k+2}}{2^{k+1}(k+1)!} t^{2k+2} && \text{(minus sign moved inside sum)} \\
&= \sum_{k=0}^{m} \frac{(-1)^{k+2}}{2^{k+1}(k+1)!} t^{2k+2}.
\end{aligned}$$

Let us now introduce the new summing variable $k' := k + 1$. The sum above then becomes

$$\sum_{k'=1}^{m+1} \frac{(-1)^{k'+1}}{2^{k'}(k')!} t^{2k'}.$$

But this is exactly the right hand side of (1) for $n = m + 1$, hence we have shown that (1) holds for $n = m + 1$, and by induction for any $n = 1, 2, \ldots$.

c) We know that $C[0, 1]$ is a Banach space with the norm $\|\cdot\|_\infty$. Furthermore F is a contraction, since

$$\begin{aligned}
\|Fx - Fy\|_\infty &= \|\frac{t^2}{2} - (Gx)(t) - \left(\frac{t^2}{2} - (Gy)(t)\right)\|_\infty \\
&= \|Gx - Gy\|_\infty \\
&< K\|x - y\|_\infty
\end{aligned}$$

by a). Hence Banach's fixed point theorem applies, and F has a unique fixed point x^*. We know from the course notes (see corollary 3.19) that for any starting point $x_0 \in C[0,1]$, the sequence $F^n x_0$ will converge to x^* as $n \to \infty$. Let us pick $x_0(t) = 0$. Then part b) shows that x^* is given pointwise fo $t \in [0,1]$ by

$$\begin{aligned}\lim_{n\to\infty} F^n x(t) &= \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k+1} \frac{t^{2k}}{2^k k!} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^{2k}}{2^k k!} \\ &= 1 - e^{-t^2/2},\end{aligned}$$

where the last step follows from recognising the Taylor series for $1 - e^{-t^2/2}$.

4 Consider the integral equation

$$f(x) = \sin x + \lambda \int_0^3 e^{-(x-y)} f(y) dy$$

for some scalar λ.

a) Determine for which λ there exists a continuous function f on $[0,3]$ that solves this integral equation.

b) Pick one of the values of λ found in **a)**. Use the method of iteration, as described in Banach's fixed point theorem, to find approximations f_1 and f_2 to a potential solution by starting with $f_0(x) = 1$ on $[0,3]$.

Solution. **a)** We will use theorem 3.20 in the lecture notes. Some of the conditions for this theorem will always be satisfied:

- $\sin(x)$ is continuous on $[0,3]$ (this is g in the notation of 3.20)
- $e^{-(x-y)}$ is continuous on $[0,3] \times [0,3]$ (k in 3.20).

But we also need that $|\lambda| < \frac{1}{3\|e^{-(x-y)}\|_\infty}$ for the solution to exist, by the same theorem. We easily find that

$$\|e^{-(x-y)}\|_\infty = \sup_{x,y\in[0,3]} |e^{-(x-y)}| = e^3.$$

(This is easy to see, since $e^{-(x-y)} = e^{-x}e^y$, so we just need to find the supremum of each factor and multiply them.). We conclude that a solution exists for $|\lambda| < \frac{1}{3e^3} \approx 0.017$.
b) We pick $\lambda = 1/100$. With $f_0(x) = 1$ for $x \in [0,3]$, we find by iteration that

$$f_1(x) = \sin(x) + 0.01 \int_0^3 e^{-(x-y)}\ dy = \sin(x) + 0.01(e^3 - 1)e^{-x}.$$

We insert this back into the iteration once again, to obtain

$$\begin{aligned}
f_2(x) &= \sin(x) + 0.01 \int_0^3 e^{-(x-y)} \left(\sin(y) + 0.01(e^3 - 1)e^{-y}\right) \, dy \\
&= \sin(x) + 0.01 \left(0.01(e^3 + e^{-3} - 2) + \frac{1}{2}e^{-x}(1 + e^3 \sin(3) - e^3 \cos(3))\right) \\
&= C + \sin(x) + Ke^{-x}
\end{aligned}$$

where C, K are constants that you may calculate. It is not difficult to see that the function f_n after n iterations will also be of the same form, namely

$$f_n(x) = C_n + \sin(x) + K_n e^{-x}.$$

5 Apply Picard iteration to

$$x'(t) = 1 + x^2, \quad x(0) = 0.$$

Find x_3 and the exact solution (notice that the equation is separable), and show that the terms involving $t, t^2, \ldots, t^5$ in $x_3(t)$ are the same as those of the Taylor series of the exact solution.

Solution. In the notation of section 3.5.2 in the notes, we have that $f(t, x) = 1 + x^2$, $t_0 = 0$ and $x(t_0) = x(0) = 0$. By corollary 3.22 Picard iteration is therefore given by $x_0 = x(0) = 0$ and

$$x_{n+1}(t) = \int_0^t f(s, x_n(s)) \, ds.$$

Therefore

$$\begin{aligned}
x_1(t) &= \int_0^t (1 + x_0(s)^2) \, ds \\
&= \int_0^t 1 \, ds = t.
\end{aligned}$$

$$\begin{aligned}
x_2(t) &= \int_0^t (1 + x_1(s)^2) \, ds \\
&= \int_0^t (1 + s^2) \, ds = t + \frac{t^3}{3}.
\end{aligned}$$

$$\begin{aligned}
x_3(t) &= \int_0^t (1 + x_2(s)^2)\ ds \\
&= \int_0^t \left(1 + \left(s + \frac{s^3}{3}\right)^2\right) \\
&= \int_0^t \left(1 + s^2 + \frac{s^6}{9} + \frac{2s^4}{3}\right)\ ds \\
&= t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63}.
\end{aligned}$$

The exact solution is the function $x(t)$ satisfying

$$\frac{dx}{dt} = 1 + x^2.$$

As you once learned in an elementary course, such equations can be solved by separation of variables:

$$\frac{dx}{1 + x^2} = dt,$$

and by integrating both sides of this equation we find that

$$\arctan x = t + C$$

for some constant C, which means that $x(t) = \tan(t + C)$. Since $x(0) = 0$ we see that $C = 0$, hence the solution is

$$x(t) = \tan(t).$$

The Taylor series of tan centered at $t = 0$ is (you may look it up)

$$\tan(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \dots$$

We observe that the terms of order up to t^5 agree with the solution we found using Picard iteration.

Remark: We did not ask you to show that the conditions in Picard-Lindelöf and Picard iteration were satisfied. Let us briefly indicate why they actually are satisfied. Pick some rectangle R containing our initial value, for instance $R = [-10, 10] \times [-10, 10]$. Then $f(t, x) = 1 + x^2$ is clearly continuous on R. To see that the Lipschitz condition is satisfied, note that

$$\begin{aligned}
|f(t, x) - f(t, y)| &= |1 + x^2 - (1 + y^2)| \\
&= |x^2 - y^2| \\
&= |x + y||x - y|.
\end{aligned}$$

When $x, y \in [-10, 10]$, the expression $|x + y|$ is bounded from above by the constant 20. This means that

$$\begin{aligned}
|f(t, x) - f(t, y)| &= |x + y||x - y| \\
&\leq 20|x - y|,
\end{aligned}$$

so the Lipschitz condition is satisfied.