Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2018

Exercise set 1: Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let X, Y and Z be sets.

a) Show that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

b) Show that $X \backslash (Y \cup Z) = (X \backslash Y) \cap (X \backslash Z)$.

Solution. a) We want to show that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$, and it is enough to show that $x \in X \cap (Y \cup Z) \iff x \in (X \cap Y) \cup (X \cap Z)$. We show this by the following chain of equivalences:

$$\begin{aligned} x \in X \cap (Y \cup Z) &\iff x \in X \text{ and } x \in Y \cup Z && \text{by the definition of } \cap \\ &\iff [x \in X \text{ and } x \in Y] \textbf{ or } [x \in X \text{ and } x \in Z] && \text{by definition of } \cup \\ &\iff x \in (X \cap Y) \cup (X \cap Z) && \text{by definition of } \cup. \end{aligned}$$

By following these equivalences, we have shown that $x \in X \cap (Y \cup Z) \iff x \in (X \cap Y) \cup (X \cap Z)$.

b) We now want to show that $X \backslash (Y \cup Z) = (X \backslash Y) \cap (X \backslash Z)$, and we will do so by showing that $x \in X \backslash (Y \cup Z) \iff x \in (X \backslash Y) \cap (X \backslash Z)$.

$$\begin{aligned} x \in X \backslash (Y \cup Z) &\iff x \in X \text{ and } x \notin Y \cup Z && \text{definition of } \backslash \\ &\iff x \in X \text{ and } x \in Y^C \cap Z^C && \text{de Morgan's law} \\ &\iff x \in X \text{ and } x \notin Y \text{ and } x \notin Z && \text{definition of } \cap \text{ and complement} \\ &\iff [x \in X \text{ and } x \notin Y] \textbf{ and } [x \in X \text{ and } x \notin Z] \\ &\iff x \in X \backslash Y \cap X \backslash Z && \text{definition of } \cap \text{ and } \backslash \end{aligned}$$

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2 Define functions on $\mathbb{R}$ with values in $\mathbb{R}$:

i) A function that is not left invertible;

ii) A function that is not right invertible.

Show that the given functions have their respective properties.

Solution. **i)** This is, by the lecture notes, the same as finding a function that is not injective. The function f defined by $f(x) = x^2$ is such a function. It is not injective, since $f(-1) = f(1) = 1$. **ii)** We need to find a function that is not surjective. The same function as before will actually work, since its image contains no negative values. A slightly more interesting example is the function $x \mapsto e^x$, which is injective yet not surjective.

3 Given the linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T = Ax$ with

$$A = \begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix}.$$

a) Show that the matrix

$$A_l^{-1} = \frac{1}{9} \begin{pmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{pmatrix}$$

induces a left inverse T_l^{-1} of T.
This left inverse is not unique. Show that

$$\frac{1}{2} \begin{pmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{pmatrix}$$

gives another left inverse.

b) Turn this example into one for right inverses. Concretely, find a mapping $S : \mathbb{R}^3 \to \mathbb{R}^2$ that is based on the mapping T and give a right inverse for this mapping.

Solution.
a) A_l^{-1} "induces a left inverse T_l^{-1} of T" if we define $T_l^{-1}y = A_l^{-1}y$ for $y \in \mathbb{R}^3$. To check that this is indeed a left inverse, we need to check that $T_l^{-1}Tx = x$ for any $x \in \mathbb{R}^2$. By the definitions of the mappings, we need to check that $A_l^{-1}Ay = y$ for any $y \in \mathbb{R}^3$, or, equivalently, that $A_l^{-1}A$ is the identity matrix:

$$A_l^{-1}A = \frac{1}{9} \begin{pmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{pmatrix} \begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly we can show that the other matrix gives a left inverse, since

$$\frac{1}{2} \begin{pmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) The simplest way of finding such an operator S and a right inverse S_r^{-1} is to exploit some properties of the transpose of matrices. We know from linear algebra that if X and Y are matrices such that the matrix product XY is defined, then $(XY)^T = Y^T X^T$. In the previous problem we found that $A_l^{-1}A = I$, where I denotes the identity matrix. Taking the transpose we find that $I = I^T = (A_l^{-1}A)^T = A^T(A_l^{-1})^T$. Hence, if we define S to be the mapping induced by A^T, we see that the mapping induced by $(A_l^{-1})^T$ is a right inverse of this S.

4 Show that the Cartesian product of two (infinite) countable sets is countable.

Solution. Let X and Y be two infinite, countable sets. The idea we will follow is rather simple: we know from the lecture notes that $\mathbb{N} \times \mathbb{N}$ is countable, so we will construct a bijection from $X \times Y$ to $\mathbb{N} \times \mathbb{N}$ and use this to show that $X \times Y$ is countable. By the definition of countability we can find bijections

$$\phi_X : X \to \mathbb{N}$$
$$\phi_Y : Y \to \mathbb{N}.$$

Using ϕ_X and ϕ_Y we can construct a bijection $\phi : X \times Y \to \mathbb{N} \times \mathbb{N}$ by defining

$$\phi(x, y) = (\phi_X(x), \phi_Y(y)) \quad \text{for } (x, y) \in X \times Y.$$

Let us quickly check that ϕ is a bijection:

1. If $\phi(x, y) = \phi(x', y')$, then we must have that $\phi_X(x) = \phi_X(x')$. Since ϕ_X is a bijection it is in particular injective, hence $x = x'$. By the same reasoning we must have $\phi_Y(y) = \phi_Y(y')$ and therefore $y = y'$. In conclusion $(x, y) = (x', y')$, and ϕ is injective.
2. To show surjectivity, we let $(m, n) \in \mathbb{N} \times \mathbb{N}$. Since ϕ_X is surjective, we can find $x \in X$ such that $\phi_X(x) = m$, and since ϕ_Y is surjective, we can find $y \in Y$ such that $\phi_X(y) = n$. Then $\phi(x, y) = (m, n)$, so ϕ is bijective.

From the lecture notes we know that $\mathbb{N} \times \mathbb{N}$ is countable, so there is a bijection $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. We also know from the lecture notes that the composition of two bijections is a bijection. Hence

$$\psi \circ \phi : X \times Y \to \mathbb{N}$$

is a bijection, and $X \times Y$ is countable.

5 Show that the sets $\mathbb{Z}$ of integers and $\mathbb{Q}$ of rational numbers are countable.

Solution. Let us start by showing that $\mathbb{Z}$ is countable. The quick way of solving this is to use proposition 1.3.6 in the lecture notes: countable unions of countable subsets are themselves countable. In this case $\mathbb{Z}$ is the union of three countable sets: the positive integers (countable by definition), the negative integers (obviously countable - make sure that you would know how to prove it!) and $\{0\}$ – hence $\mathbb{Z}$ is countable.

For those interested, we also solve the problem using the definition in a way that hopefully makes the result obvious. We need to find an injection φ from $\mathbb{Z}$ to $\mathbb{N}$. To construct φ, we need to assign to each integer a natural number. There is an obvious way of doing this:

Integer n	Natural number $\varphi(n)$
...	...
-3	7
-2	5
-1	3
0	1
1	2
2	4
3	6
...	...

It is not difficult to find the general formula for φ:

$$\varphi(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2|n| + 1 & \text{if } n < 0 \\ 1 & \text{if } n = 0. \end{cases}$$

We leave it to the reader to check that φ is injective - it is not difficult.

Now let us turn to $\mathbb{Q}$. Any number in $\mathbb{Q}$ can be written in a unique way as $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no common divisor (this last statement means, for instance, that we would write $\frac{1}{5}$ and not $\frac{10}{50}$).

We define a map $\varphi : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $\varphi\left(\frac{p}{q}\right) = (p, q)$. It is easy to see that φ is injective [1]. By problem (4), $\mathbb{Z} \times \mathbb{N}$ is countable, so by definition there exists an injective map $\Psi : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. Now consider the composition

$$\Psi \circ \varphi : \mathbb{Q} \to \mathbb{N}.$$

We claim that this map is injective, which would prove that $\mathbb{Q}$ is countable. In fact, any composition of two injective maps must itself be injective. Assume that

$$\Psi \circ \varphi(x) = \Psi \circ \varphi(y).$$

By definition of composition this means that

$$\Psi(\varphi(x)) = \Psi(\varphi(y)),$$

and since Ψ is injective this means that $\varphi(x) = \varphi(y)$, and the injectivity of φ now implies that $x = y$ – hence $\Psi \circ \varphi$ is injective.

[1] Make sure that you see this.