Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods

Fall 2018

Exercise set 9

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Show that point evaluation is a bounded linear functional on $C[a,b]$. That is, for some fixed $t_0 \in [a,b]$ define $f_{t_0} : C[a,b] \to \mathbb{C}$ by

$$f_{t_0}(x) = x(t_0), \quad x \in C[a,b],$$

and show that f_{t_0} is a bounded linear functional on $C[a,b]$.

2 Let T be a bounded linear operator on a Hilbert space X. Show that the operator norm of T can be expressed in terms of the inner product of X:

$$\|T\| = \sup\{\langle Tx, y\rangle : x, y \in X \text{ with } \|x\| = \|y\| = 1\}.$$

3 Let c_f be the subspace of ℓ^2 that consists of all sequences with finitely many non-zero terms.

a) Show that best approximation fails for c_f.

b) Why does this not contradict the best approximation theorem?

4 Let M be a subspace of an inner product space $(X, \langle\cdot,\cdot\rangle)$. Show that the orthogonal complement $M^\perp$ is closed.

5 Let M be the plane of $\mathbb{R}^3$ given by $x_1 + x_2 + x_3 = 0$. Find the linear mapping that is the orthogonal projection of $\mathbb{R}^3$ onto this plane.

6 (*Exam 2017, problem 4*) For $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{R})$ we define the linear operator $T_a : \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ by

$$T_a(x_1, x_2, \ldots) = (a_1x_1, 0, a_3x_3, 0, \ldots), \quad x \in \ell^2(\mathbb{R}).$$

a) Show that T_a is bounded on $\ell^2(\mathbb{R})$.

b) Determine the operator norm of T_a.

c) Show that the range of T_a is closed.

d) Determine the orthogonal complement of $\ker(T_a)$.

e) Determine for which sequences $a \in \ell^\infty(\mathbb{R})$ the operator T_a satisfies $T_a^2 = T_a$.