Norwegian University of Science and Technology
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TMA4145 Linear Methods
Fall 2018

Exercise set 12

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let

$$z_1 = \sqrt{\frac{2}{3}}\begin{bmatrix}1\\0\end{bmatrix}, \quad z_2 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\\sqrt{3}/2\end{bmatrix}, \quad z_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\-\sqrt{3}/2\end{bmatrix}.$$

Show that for every $x \in \mathbb{R}^2$ we have

a)

$$\|x\|^2 = \sum_{i=1}^{3} |\langle x, z_i\rangle|^2$$

b)

$$x = \sum_{i=1}^{3} \langle x, z_i\rangle z_i$$

Remark. The vectors z_1, z_2, z_3 span $\mathbb{R}^2$, but they are obviously not an orthonormal basis (they are not even linearly independent). Still, they satisfy a generalization of Parseval's identity and "act like" an orthonormal basis. Such systems appear very naturally in applications (e.g. in signal analysis), and are often called Parseval frames.

2 Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be equivalent norms on a vector space X. Show that any set $U \subset X$ is open in $(X, \|\cdot\|_a)$ if and only if it is open in $(X, \|\cdot\|_b)$.

Remark. This is in fact a two-way implication; if any set $U \subset X$ is open in $(X, \|\cdot\|_a)$ if and only if it is open in $(X, \|\cdot\|_b)$, then necessarily the norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent on X.

3 Suppose that $v_1, \dots, v_k$ are non-zero eigenvectors of an operator T corresponding to distinct eigenvectors $\lambda_1, \dots, \lambda_k$. Show that $\{v_1, \dots v_k\}$ is a linearly independent set.

4 Let T be the shift operator on ℓ^2 defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$.

1. Show that T has no eigenvalues.
2. Does T^* have any eigenvalues?

$\boxed{5}$ Let U be a $n \times n$ matrix with columns $u_1, ..., u_n$. Show that the following statements are equivalent:

1. U is unitary.
2. $\{u_1, ..., u_n\}$ is an orthonormal basis of $\mathbb{C}^n$.

$\boxed{6}$ Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

a) Compute the singular value decomposition of A.

b) Use the result of a) to find:

1. Bases for the following vector spaces: $\ker(A), \ker(A^*), \operatorname{ran}(A), \operatorname{ran}(A^*)$.
2. The pseudo-inverse of A.
3. Find the minimal norm solution of $Ax = b$ for $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.