Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2018

Exercise set 10

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let X be a Hilbert space and $T : X \to X$ a bounded linear operator. Suppose x and x' are two elements in X. Show that if

$$\langle x, y\rangle = \langle x', y\rangle \quad \text{for all } y \in X,$$

then $x = x'$.

2 Define on $C[0,1]$ the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$$

Show that $(C[0,1], \langle\cdot,\cdot\rangle)$ is an inner product space, but that it is not complete with respect to the norm

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$$

induced by the inner product.

3 Let X_1 and X_2 be two Hilbert spaces and $T \in B(X_1, X_2)$.

a) Show that there exists $T^* \in B(X_2, X_1)$ such that $\langle Tx, y\rangle_{X_2} = \langle x, T^*y\rangle_{X_1}$ for any $x \in X_1$, $y \in X_2$.

(Note: We treated the case $X_1 = X_2$ in class.)

b) Prove that $\ker T = \ker T^*T$.

4 Let $T : X \to X$ be a bounded linear operator on a Hilbert space X. Show that

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

$\boxed{5}$ Consider the multiplication operator T_a on $(\ell^2, \langle\cdot,\cdot\rangle)$ given by

$$T_a x = (a_j x_j)_{j\in\mathbb{N}}$$

for a fixed sequence $a = (a_j)_{j\in\mathbb{N}} \in \ell^\infty$.

a) Determine the adjoint operator T_a^*.

b) Is T_a a normal operator? Under which condition(s) on the sequence a is T_a unitary; self-adjoint?

$\boxed{6}$ Let M be a closed subspace of a Hilbert space X, which by the projection theorem is given by the direct sum

$$X = M \oplus M^\perp.$$

Show that the projection onto M is self-adjoint.